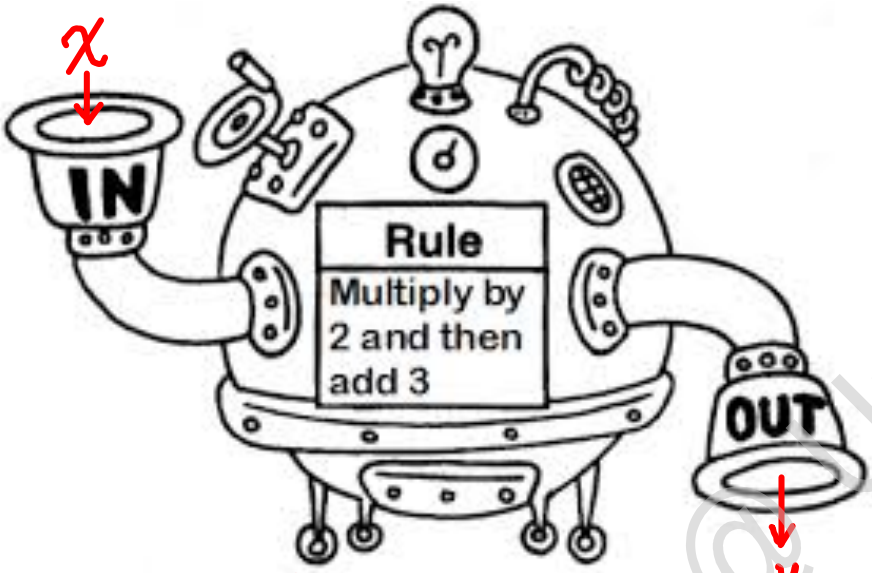
§2.1-2.5

Functions and Graphs of Functions

What is a mathematical function ?



input x	output y
-1	1
0	3
1	5

pair (x, y)

$(-1, 1)$
$(0, 3)$
$(1, 5)$

* A function : A correspondence / relation / mapping that assigns **ONE** output to each input according to a rule.

Notations : $f(x) = 2x + 3$

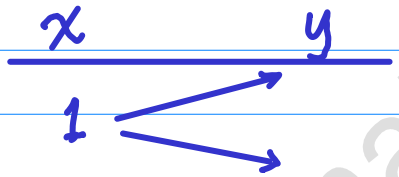
- f : name of the function
- x : variable for the input
- $2x + 3$: rule to produce the output

or $y = 2x + 3$

- y : output
- $2x + 3$: rule to produce output from the input x

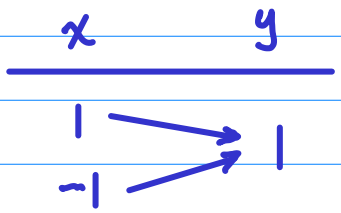
Notes: • Exactly ONE output corresponds to each input.

E.g. relation $|y| = x$, where x is input and y is output, is NOT a function. (why?)



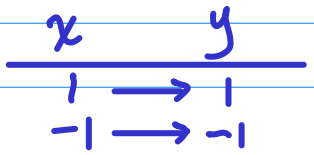
• Different inputs may lead to the same output.

E.g. relation $y = |x|$, where x is input and y is output, is a function



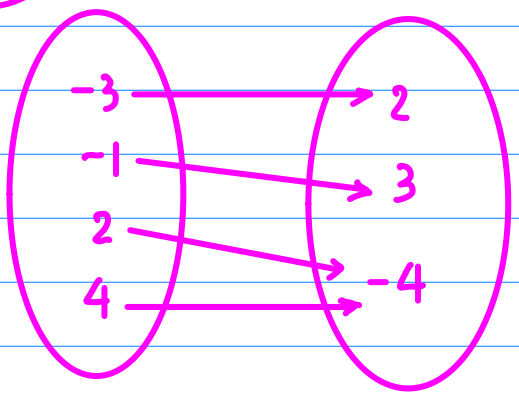
• If any two different inputs lead to two different outputs, then the function is said to be one-to-one

E.g. $y = x$

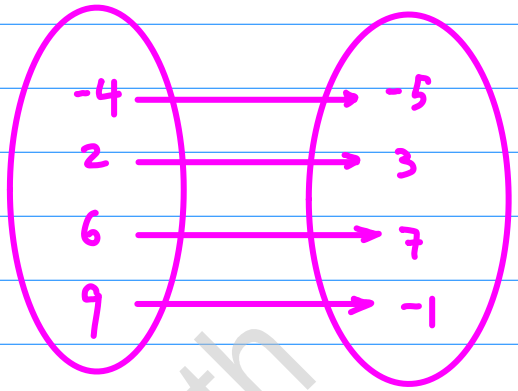


poll Which relation below is NOT a function ?

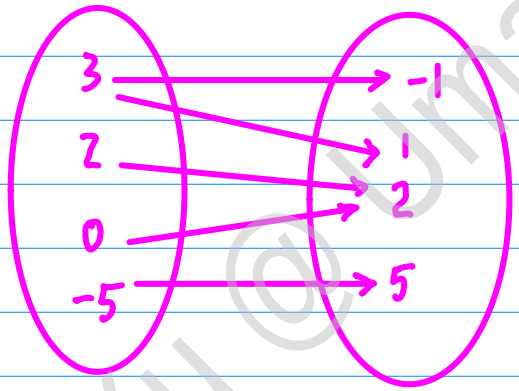
(a)



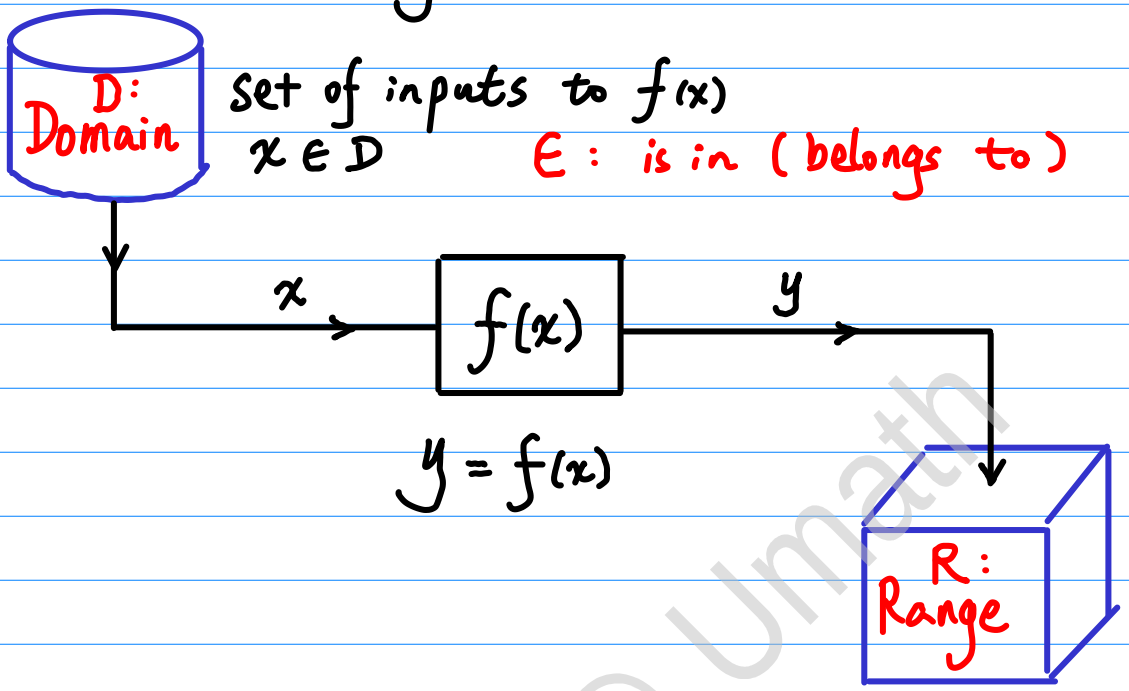
(b)



(c)



* Domain and Range



Set of inputs to $f(x)$
$x \in D$ $\in$: is in (belongs to)

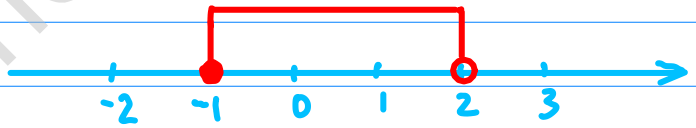
$y = f(x)$

Set of outputs from $f(x)$
$f(x) \in R$

E.g. $f(x) = -2x + 3$, $-1 \leq x < 2$

The specified domain : $D = [-1, 2)$

$[-1, 2)$: an interval of real numbers x, $-1 \leq x < 2$



The range :

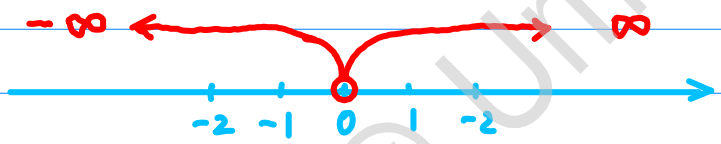
$$-1 \leq x < 2$$
$$2 \geqslant -2x > -4$$
$$-4 < -2x \leq 2$$
$$-1 < \underbrace{-2x + 3}_{f(x)} \leq 5$$

So range $R = (-1, 5]$

Note: If the domain of the function $y = f(x)$ is not specified, then the implied domain of the function f is the set of all allowed inputs.

E.g. $f(x) = \frac{1}{x}$: a real function.
↳ input & output : real

The domain : all real values except ___



The domain : $D = (-\infty, 0) \cup (0, \infty)$
↓ union (or)

Poll What is the domain of the function

$$f(x) = \sqrt{2-x} + \frac{1}{\sqrt{x-1}}$$

Note: $\cap$
↓ intersection (and)

Why do we need functions ?

$y = f(x)$: the relation between quantities x and y.

E.g. The area A of a circle with the radius r.
The relation between A and r :

$$A = f(r), \text{ where } f(r) = \pi r^2$$

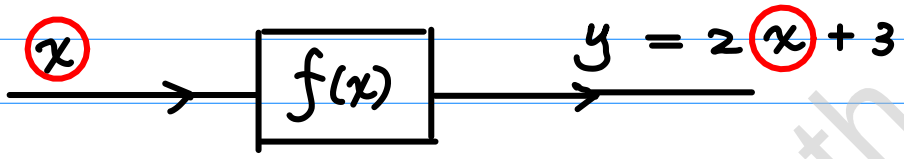
or $A = \pi r^2$

poll If a rope of the length 10 is to form a rectangle with one side length x. Write down the area y of the rectangle as a function of x. What is the domain of the function.

Evaluating Functions and Solving Function Equations

* Evaluation of Functions

E.g. $f(x) = 2x + 3$



$x = 1$ $\quad y = 2 \cdot 1 + 3$

$x = 2$ $\quad y = 2 \cdot 2 + 3$

$x = 3+4$ $\quad y = 2 \cdot (3+4) + 3$

($3+4$ = 7)

Note: Treat x as a place holder !

$f(x) = 2x + 3$

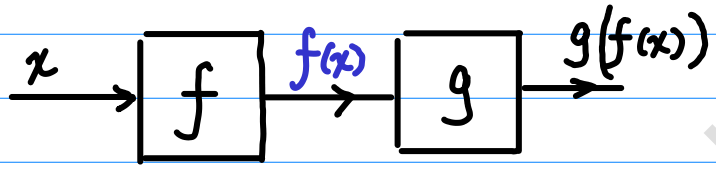
$f(\bigcirc) = 2\bigcirc + 3$

number or expression (which represents a number)

So $f(\boxed{x^2+x-1}) = 2\boxed{(x^2+x-1)} + 3$

poll $f(x^2) = 2x^4 + 3$, $f(x) = ?$

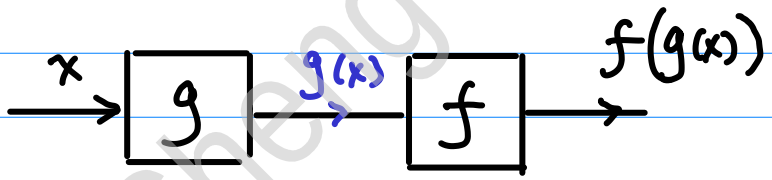
* Composite functions



$g \circ f = g(f(x))$

E.g. $f(x) = 2x + 3$, $g(x) = x^2$

$g(f(x)) = (f(x))^2 = (2x+3)^2$



$f \circ g = f(g(x))$

E.g. $f(x) = 2x + 3$, $g(x) = x^2$

$f(g(x)) = 2g(x) + 3 = 2x^2 + 3$

poll $f(x) = 2x+3$, $f(f(-x)) = ?$

* Function Equations:

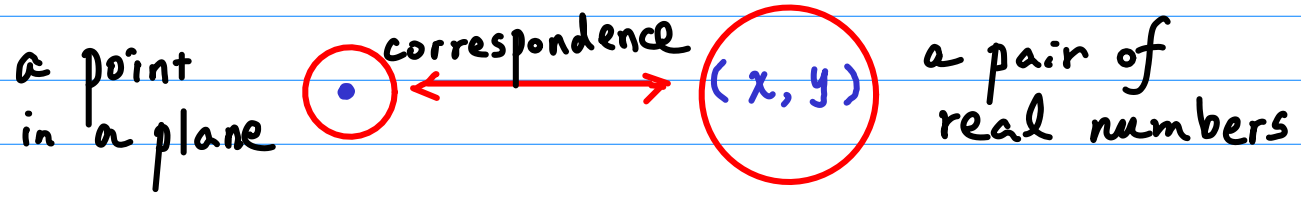
E.g. $f(x+y) = x + f(y)$ for any x and y.
If $f(0) = 1$, then $f(2020) = ?$

[Soln.:] Let $y = 0$ in $f(x+y) = x + f(y)$
Then $f(x+0) = x + f(0)$
$f(x) = x + 1$
So $f(2020) = 2020 + 1 = 2021$

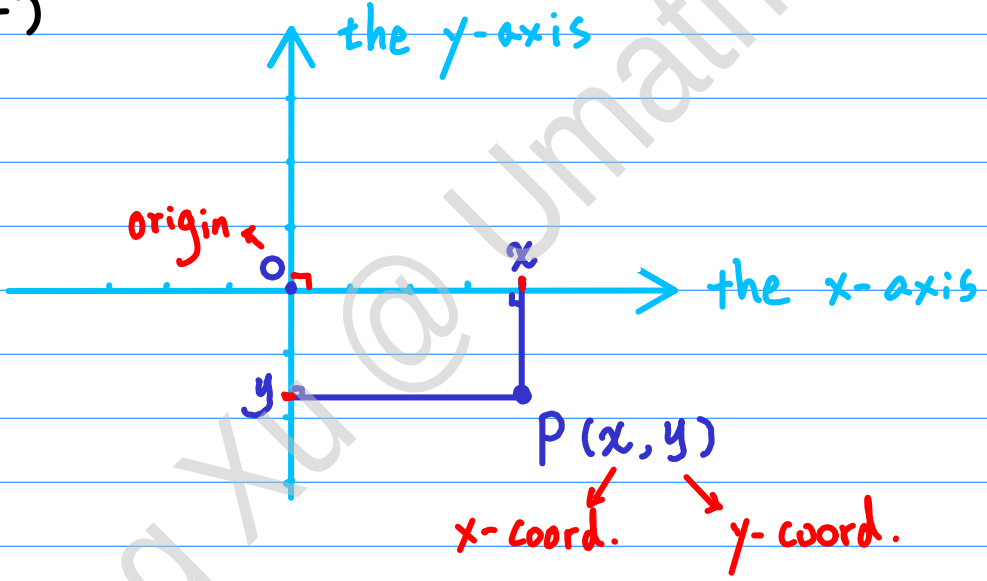
Remark: Try to substitute various values or expressions for the variables in the function equation with given conditions and needed solutions in mind.

poll Find $f(x)$ if $f(x+y) - f(x-y) = 4xy$ for all x, y and $f(0) = 1$.

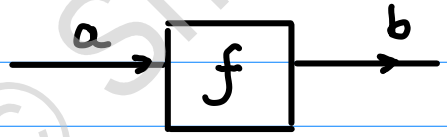
Graphs of Functions



* Cartesian Coordinate System
(rectangular)

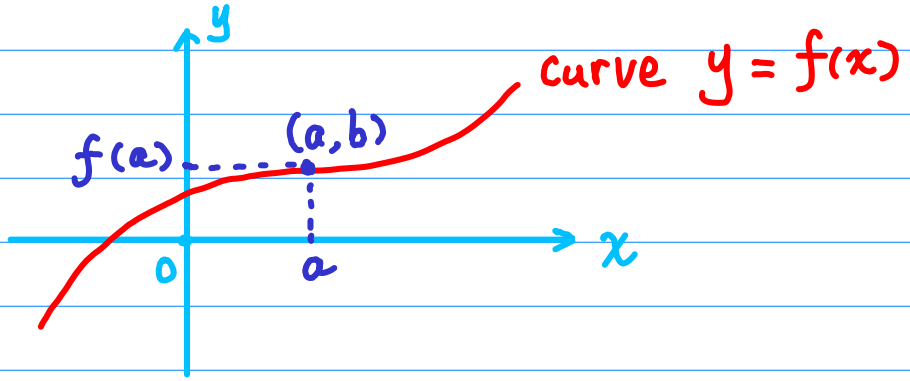


* Graph of the function $y = f(x)$

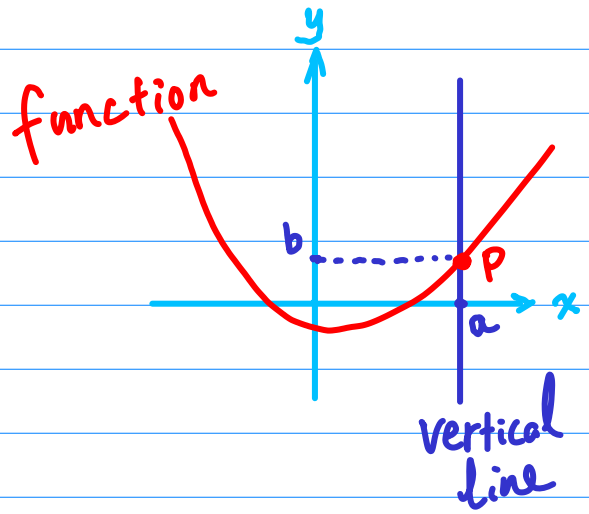


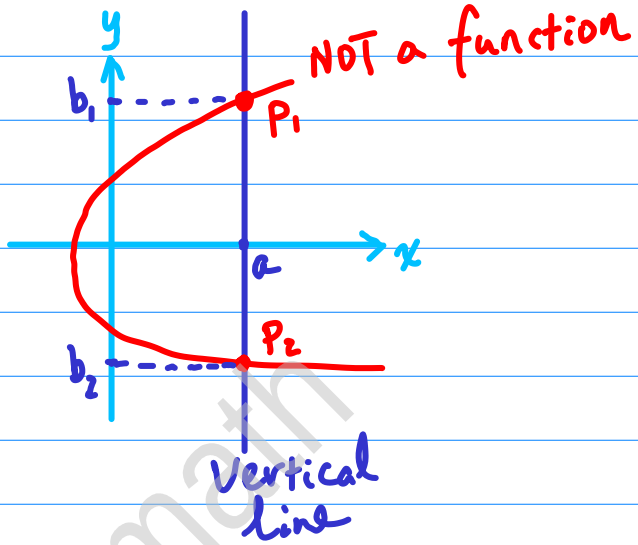
: $(a, b) \leftrightarrow$ a point

$b = f(a)$



- Vertical line test



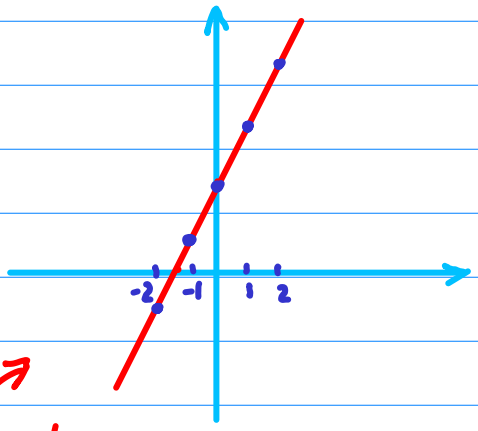


Why? A function: ___ output assigned to each input

- Sketch the graph

E.g. $f(x) = 2x + 3$

x	$y = 2x+3$
-2	-1
-1	1
0	3
1	5
2	7



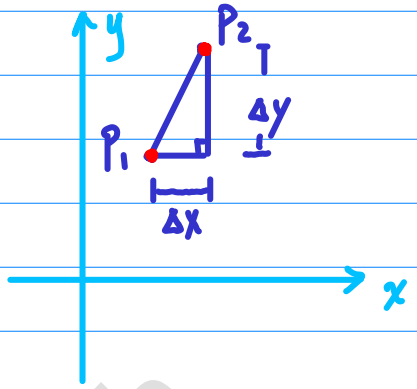
↖ how computers/calculators do

(later) Analytical approach: algebra (& calculus)

Linear Functions and Lines

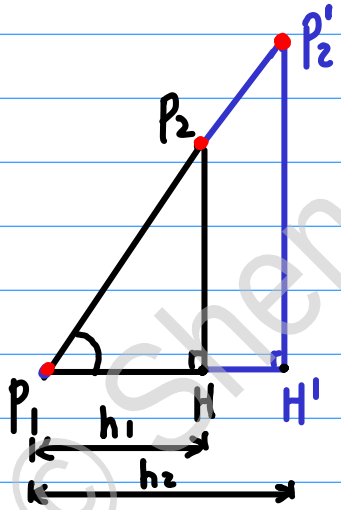
E.g. $y = 2x + 3$

point	x	y
P_1	a	$2a+3$
P_2	$a+h$	$2(a+h)+3$



	change in x	change in y
$P_1 \to P_2$:	$\Delta x = (a+h) - a = h$	$\Delta y = [2(a+h)+3] - [2a+3] = 2a+2h-2a = 2h$

$$\frac{\Delta y}{\Delta x} = \frac{2h}{h} = 2,$$ a constant indep. of h



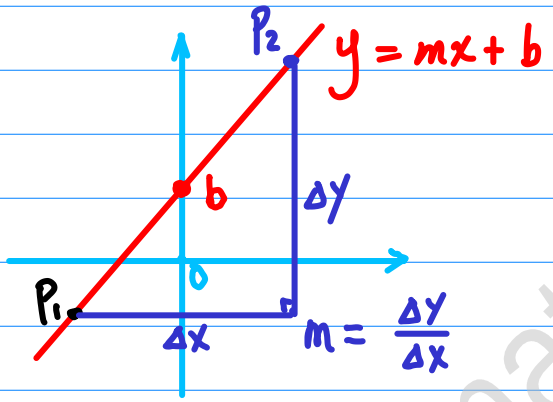
$\Delta P_1 P_2 H \sim \Delta P_1 P_2' H'$

So any three points P_1, P_2, P_2' are on a line (colinear)

Remarks:
- Graph of $y = 2x+3$ is a LINE !
- Constant $\frac{\Delta y}{\Delta x}$: the rate of change (of y with respect to x); the rate the line rises or falls; the slope of the line

* slope-intercept form of a line

$$y = mx + b$$

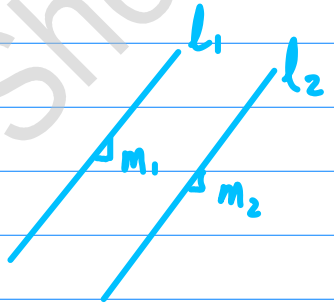


- $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ on the line

  slope $m = \dfrac{\Delta y}{\Delta x} = \dfrac{y_2 - y_1}{x_2 - x_1}$

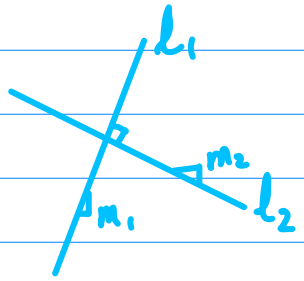
  y-intercept b : $(0, b)$ = intersection of line and y-axis

- Parallel and perpendicular lines



$l_1 \parallel l_2 \iff$

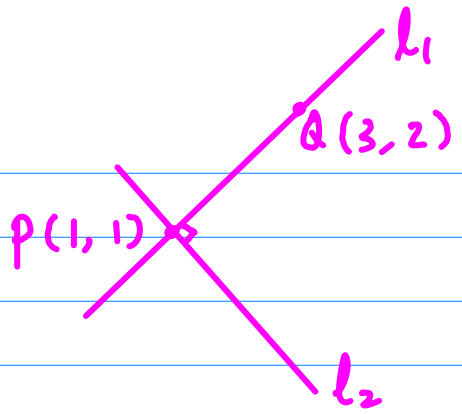
$m_1 = m_2$



$l_1 \perp l_2 \iff$

$m_1 m_2 = -1$

Poll



What is the slope of l_2?

Note: If $b = 0$, then

$y = mx$: direct variation between x, y;
(y is proportional to x)
m is constant of variation / proportion

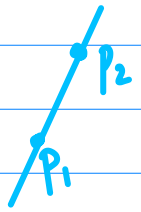
* Other forms of a Line

- point-slope form : $y - y_1 = m(x - x_1)$

where m = slope
(x_1, y_1) = a point on the line

- point-point form : $y - y_1 = \boxed{\frac{y_2 - y_1}{x_2 - x_1}}(x - x_1)$

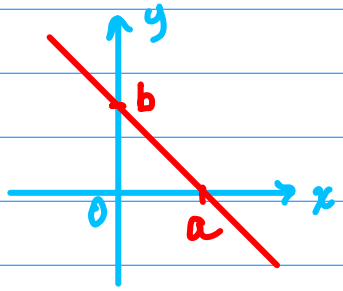
m : slope



where (x_1, y_1), (x_2, y_2) : two points on the line

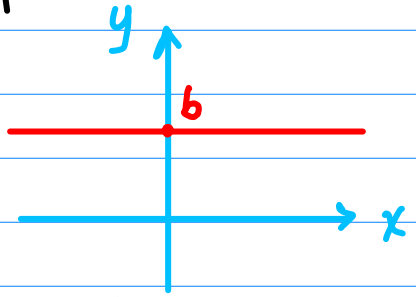
- intercept form : $\frac{x}{a} + \frac{y}{b} = 1$

x-intercept = $(a, 0)$
y-intercept = $(0, b)$

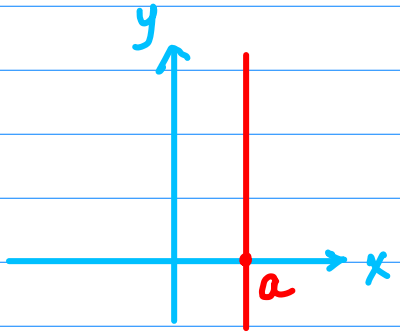


- standard form:
$Ax + By + C = 0$

- special lines



$y = b$: slope $m = 0$
Constant function
$f(x) = b$



$x = a$: slope undefined
NOT graph of a function