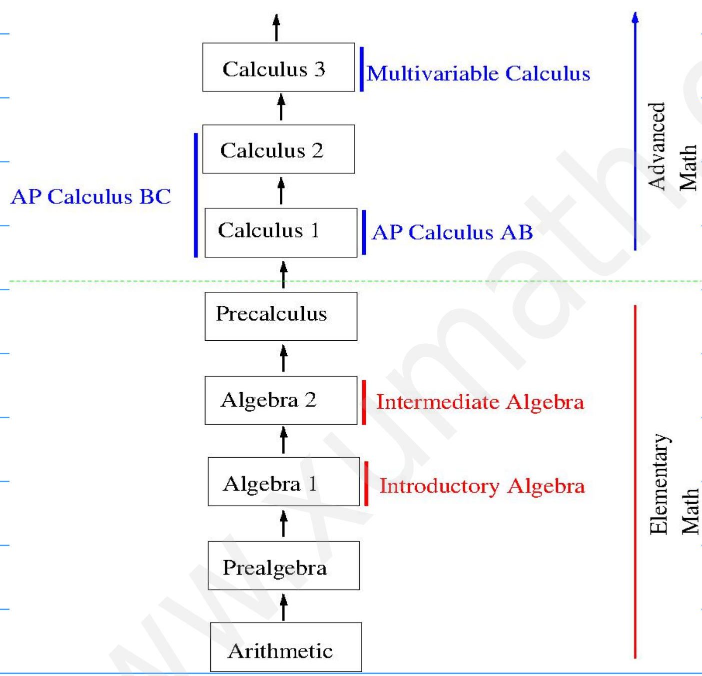
§2.1 **A Preview of Calculus** 1

What is calculus?

* An advanced math subject fundamental to STEM (Science, Technology, Engineering, Math)

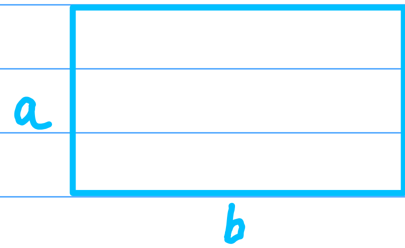


* About what?

- Instantaneous rate of change (Differentiation calculus)
- Accumulation of changing quantities (Integration calculus)

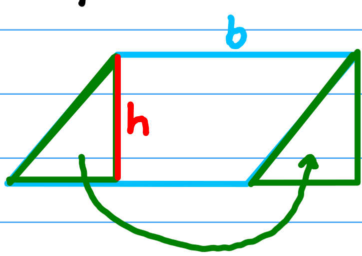
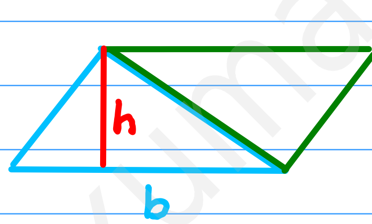
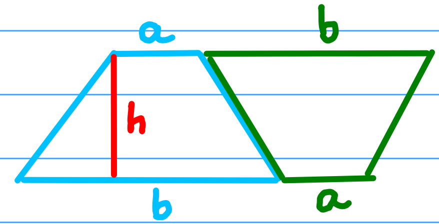
The Area Problem

* Area of a shape formed by straight lines



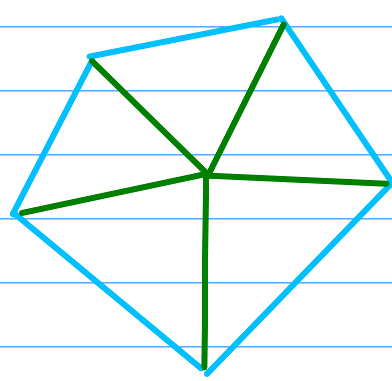
Area of a rectangle
$A = ab$

$\Downarrow$

parallelogram	triangle	trapezoid
		
$A = bh$	$A = \frac{1}{2}bh$	$A = \frac{1}{2}(a+b)h$

$\Downarrow$

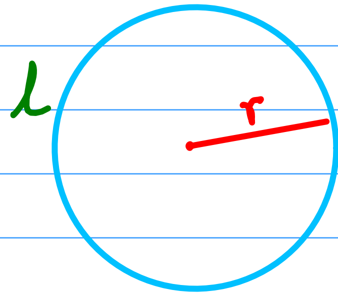
polygon



A = sum of triangles

* Area of a shape formed by curves

- Area of a circle



$$A = \pi r^2$$

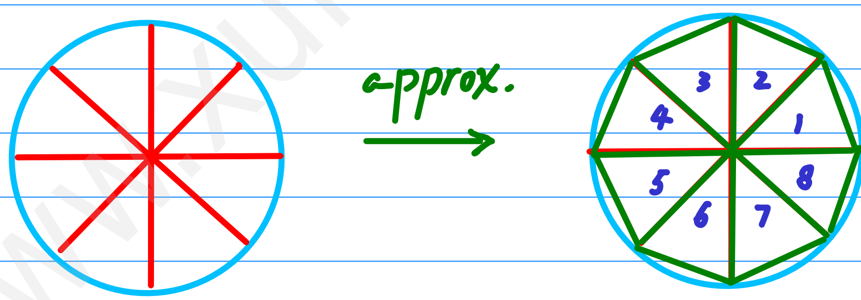
π: ratio of circumference to diameter

i.e. circumference $l = \pi \cdot 2r$

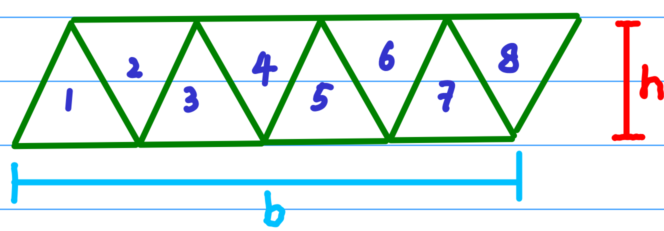
Why $A = \pi r^2$?

(1) Divide a circle into n equal sectors, approximate each sector as a triangle

$n = 8$:



(2) Approximate the area of the circle by the sum of the areas of the triangles

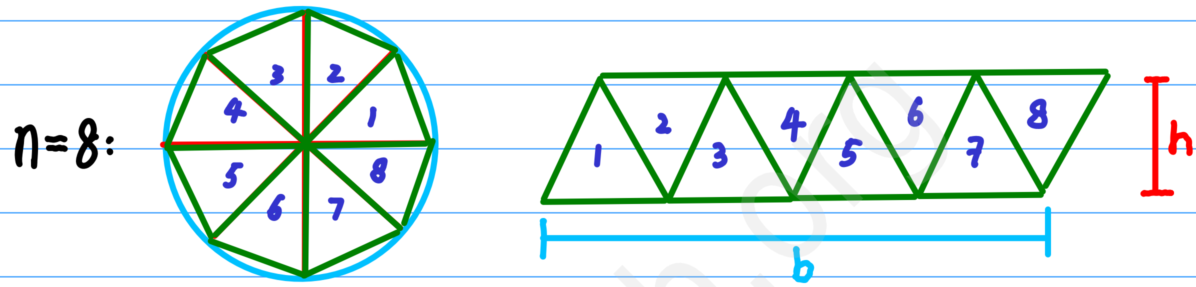


$$A \approx bh$$

(3) The Limit process : Let n approach ∞

As $n \to \infty$: the approximation $b \cdot h$ approaches the true area of the circle.

approaches



poll As $n \to \infty$, we have

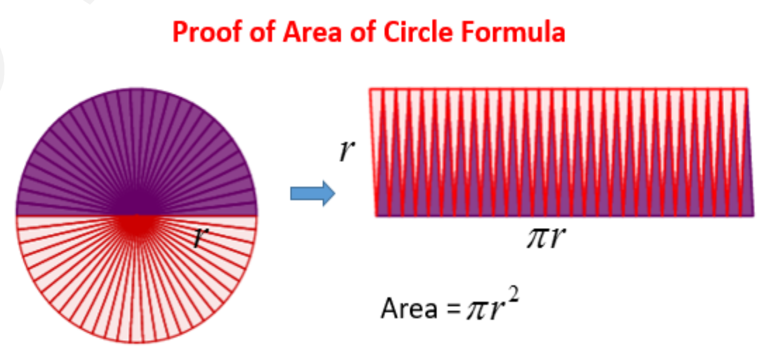
A. $b \to 2\pi r$, $h \to r$

B. $b \to \pi r$, $h \to r$

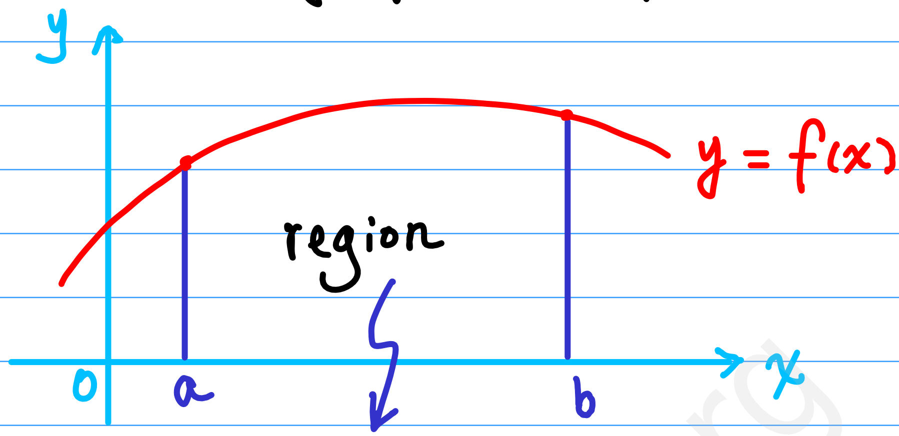
C. $b \to \pi r$, $h \to 2r$

So the area of the circle :

$$A = \pi r \cdot r = \pi r^2$$



- Area under the graph of a function

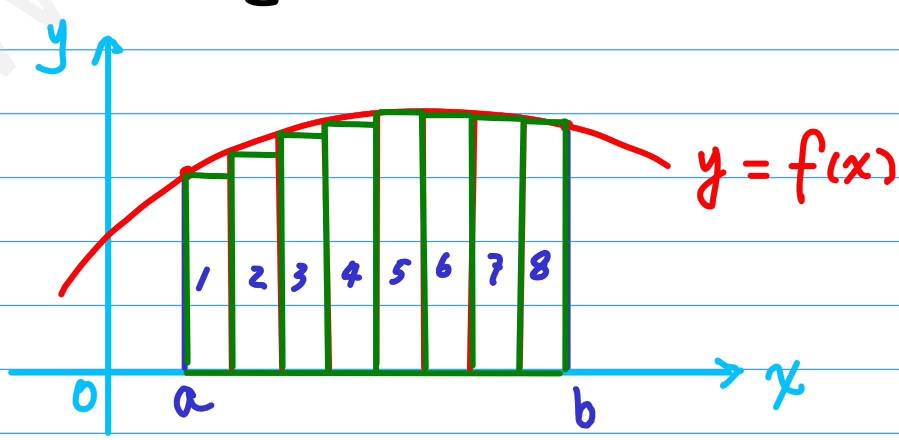


under the curve $y = f(x)$ and
above the x-axis
between the lines $x = a$, $x = b$

What is the area of the region ?

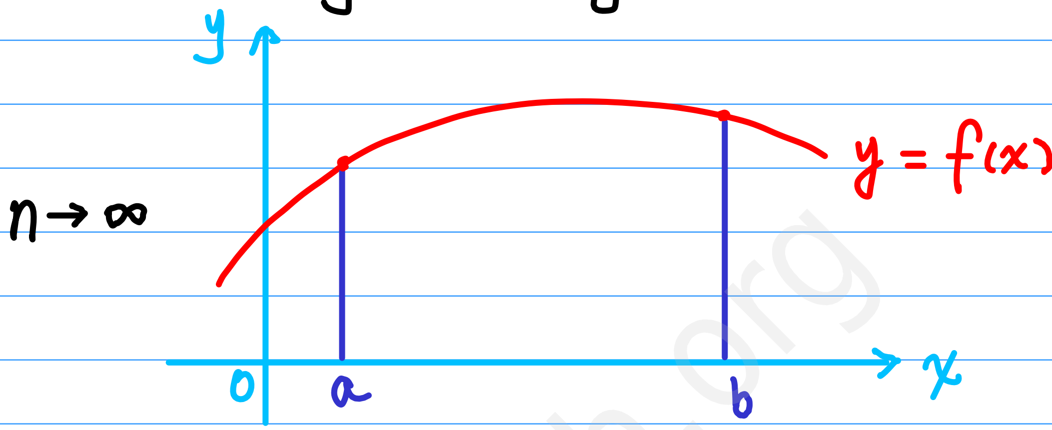
(1) Divide the region into n strips with same width ; Approximate each strip as a rectangle.

$n = 8$



(2) Approximate the area of the region by the sum of the areas of the rectangles

(3) The limit process : Let $n \to \infty$
The approximation approaches the area of the region.



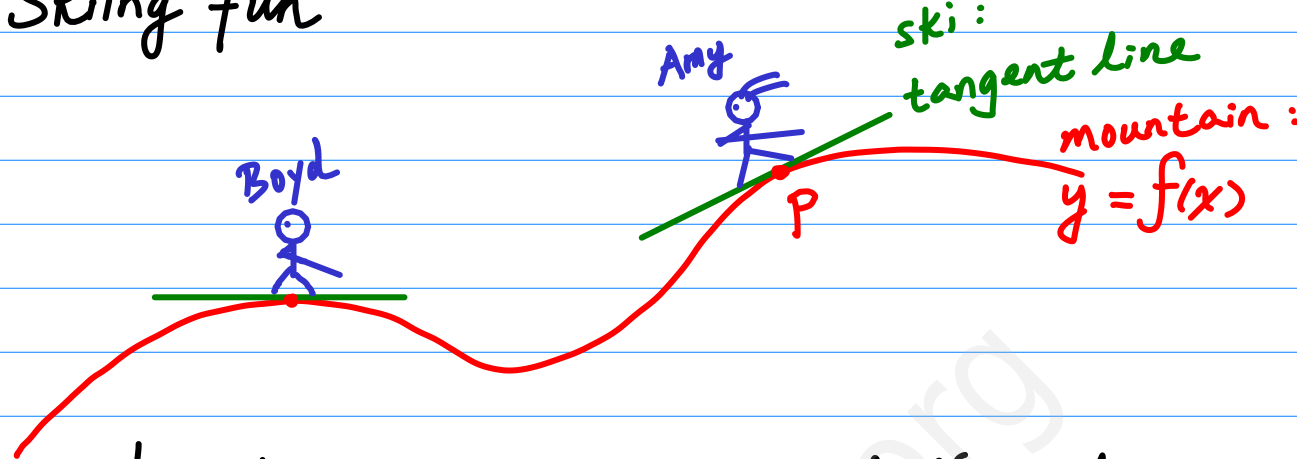
* Summary of the area problem

Integration calculus :

- divide and sum
- the limit process

The Tangent/Velocity Problem

* Skiing fun

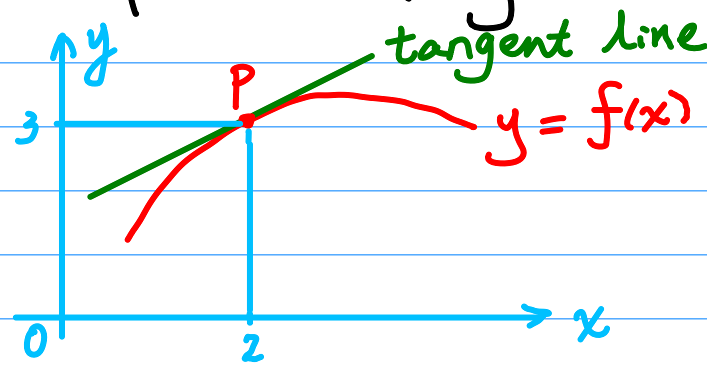


- The slopes (steepness) at different points on the curve $y = f(x)$ can be different.
- The slope of the tangent line to the curve $y = f(x)$ at a point is the slope of the curve at the point.

(OR: The tangent line has the same direction as the curve at the point.)

- Question: How to find the tangent line?

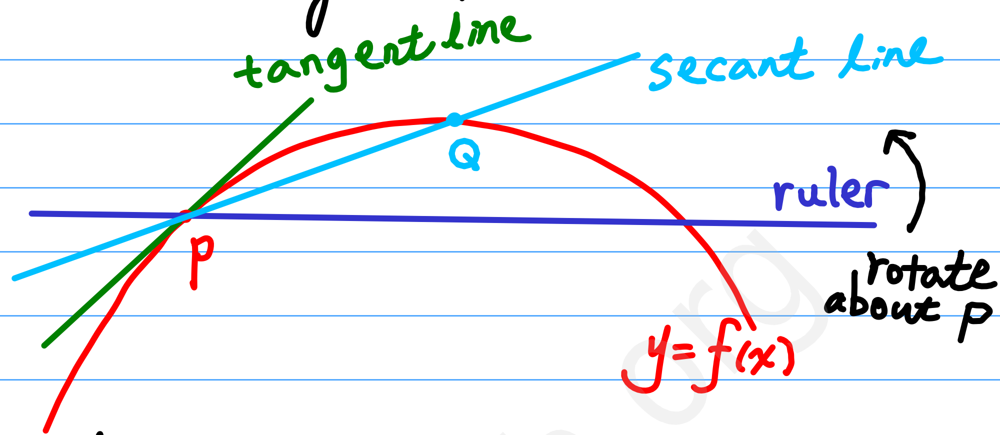
E.g.



tangent line : $\underbrace{\text{point}}_{P(2,3)}$-$\underbrace{\text{slope}}_{m\,=\,?}$ form

$$(y-3) = \underset{?}{m}(x-2)$$

- Answer : How do we draw a tangent line to a curve at a point using a ruler ?



The limit process :

The ruler is rotated about P s.t. point Q → point P.

As Q → P the secant line PQ approaches the tangent line at P.

slope of the tangent line m :

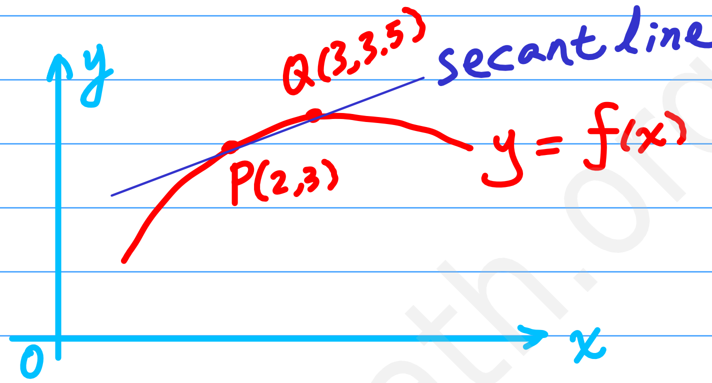
$$\underbrace{m_{PQ}}_{\text{slope of secant}} \rightarrow m \quad \text{as } Q \rightarrow P$$

poll

Find the secant line through the points $P(2,3)$, $Q(3,3.5)$ on the curve $y=f(x)$.

A. $y-3=\frac{1}{2}(x-2)$ B. $y-3=2(x-2)$

C. $y-3=-\frac{1}{2}(x-2)$

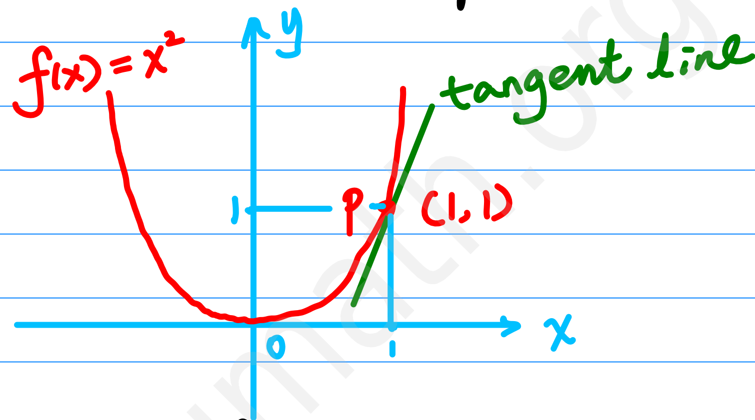


Definition: The slope of a secant line through $P(a, f(a))$ and $Q(b, f(b))$ $(b \neq a)$ on the curve $y=f(x)$ is

$$m_{PQ}=\frac{f(b)-f(a)}{b-a}$$

- An example:

Estimate the slope of the tangent line to the curve $f(x) = x^2$ at $x = 1$ using the slopes of secant lines. Write the equation of the tangent line using the estimated slope.



[Soln.:] Let $Q(x, f(x)) = (x^2, x^2)$ be another point on the curve $f(x) = x^2$.
We let $Q \to P$ by $x \to 1$ and compute slopes of secant lines m_{PQ}:

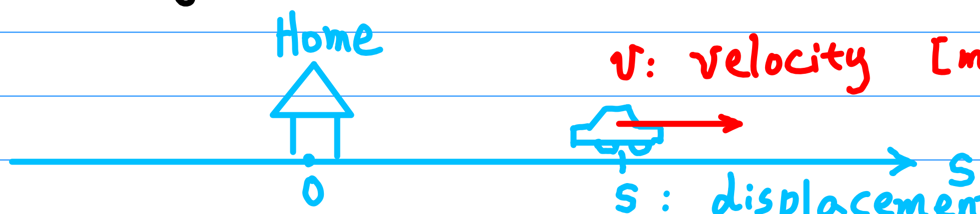
	x	$y = x^2$	$Q(x, f(x))$	m_{PQ}
Q at left	0.9	0.81	(0.9, 0.81)	$-0.19/(-0.1) = 1.9$
	0.99	0.9801	(0.99, 0.9801)	1.99
Q at right	1.1	1.21	(1.1, 1.21)	$0.21/0.1 = 2.1$
	1.01	1.0201	(1.01, 1.0201)	2.01

$$P(a, f(a)) = (1, 1) \; ; \; m_{PQ} = \frac{f(x) - f(a)}{x - a}$$

Guess: as $Q \to P$, $m_{PQ} \to 2$, so $m = 2$

tangent line: $(y - 1) = 2(x - 1)$

* Travelling fun



v: velocity [mi/h]

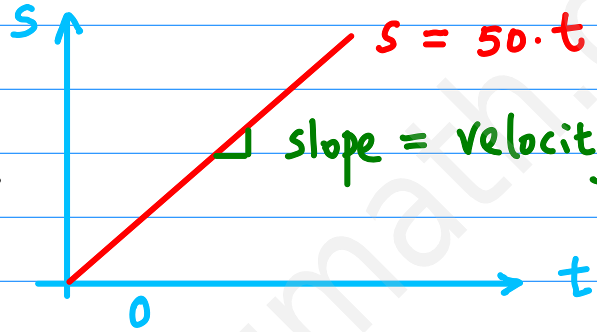
s : displacement [mi]

t: time [h]

• If velocity = constant : (Parents drive.)

E.g.

$v = 50$:

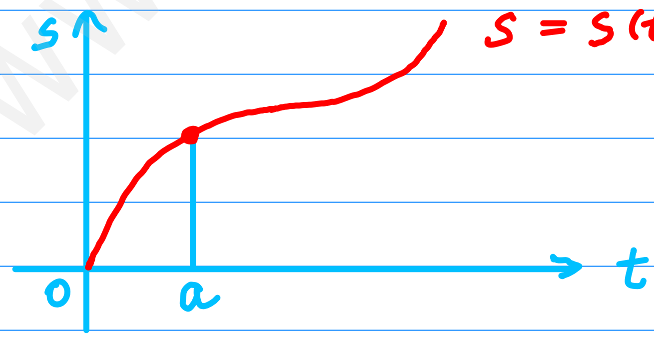


$s = 50 \cdot t$: line

slope = velocity = 50

• If the displacement is not a linear function of time : (You drive.)

E.g.

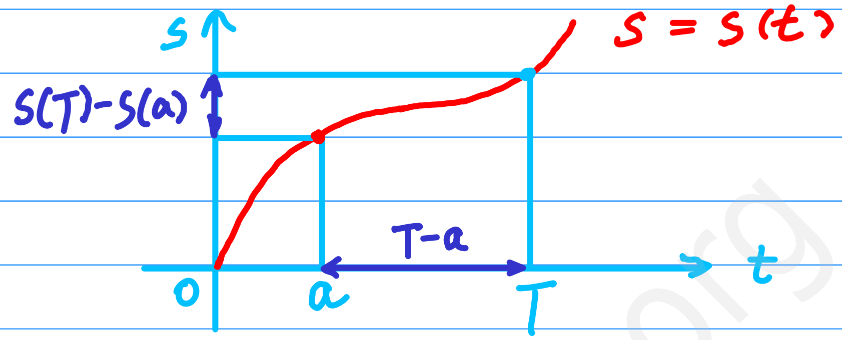


$s = s(t)$

Question : Your velocity is changing with time (NOT constant). What is your velocity at the moment/instant $t = a$?

i.e. Find the instantaneous velocity at $t = a$.

Answer : Find the average velocity between the instant a and another instant $T\ (\neq a)$.



The limiting process :
Let the other instant T approach the instant a. i.e. $T \to a$
Then the average velocity between a and T approaches the instantaneous velocity at a.

poll What is the average velocity between the instants a and T if the displacement depends on time t as the function $s(t)$?

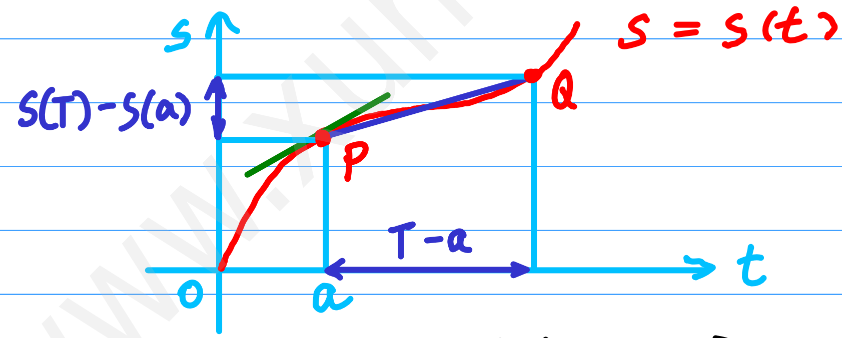
A. $\dfrac{s(T)-s(a)}{a-T}$ B. $\dfrac{T-a}{s(T)-s(a)}$

C. $\dfrac{s(T)-s(a)}{T-a}$

Definition: Let $s(t)$ be the displacement (or the position function) of an object moving along a straight line at time t. The average velocity of the object between time a and time T is

$$\boxed{v_{ave} = \frac{s(T)-s(a)}{T-a}}$$

As $T \to a$: v_{ave} approaches the instantaneous velocity at instant a.



Note: • $v_{ave} = \dfrac{s(T)-s(a)}{T-a}$ is the slope of the secant line through $P(a, s(a))$ and $Q(T, s(T))$ on the curve $s = s(t)$

• The instantaneous velocity at $t=a$ is the slope of the tangent line to the curve $s = s(t)$ at $t = a$.

* Summary of the tangent/velocity problem

Differentiation calculus :

- average
- the limit process

Calculus

$$\boxed{\begin{array}{l}\bullet\ \text{Instantaneous rate of change} \\ \quad (\underline{\text{Differentiation}}\ \text{calculus}) \\ \bullet\ \text{Accumulation of changing quantities} \\ \quad (\underline{\text{Integration}}\ \text{calculus})\end{array}}$$

$$\Uparrow$$

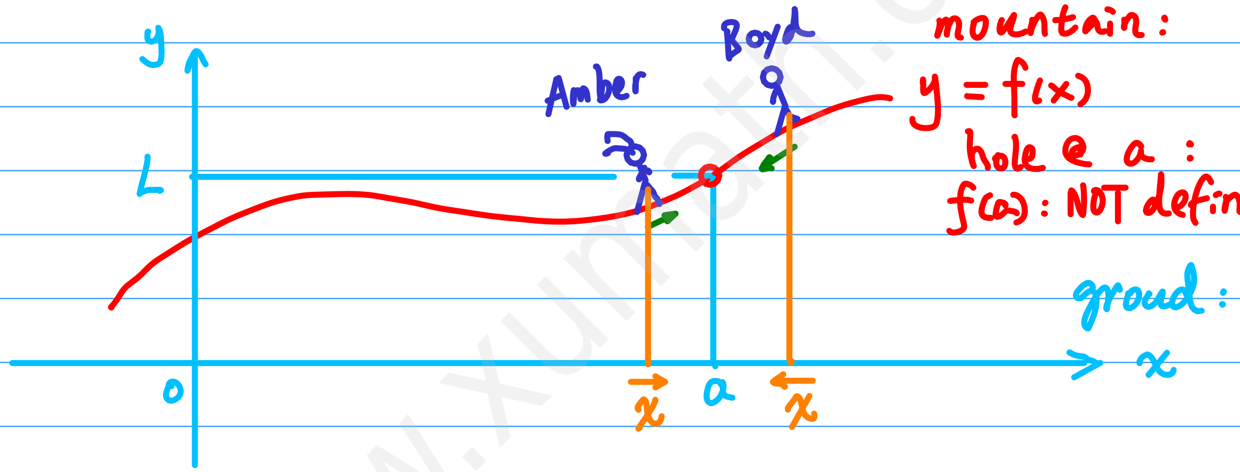
$$\boxed{\text{The limit process}}$$

§2.2

The Limit of A Function

* Climbing fun :

Amber or Boyd approach the hole .
Ground position of Amber or Boyd : x
Ground position of the hole : a
Height of Amber or Boyd : $f(x)$



poll As the ground position of Amber or Boyd x approaches a, their heights approach what ?

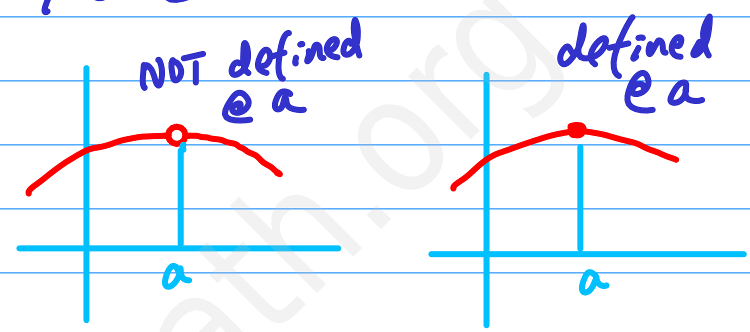
A. a B. L C. NOT defined

Intuitive Definition of a Limit

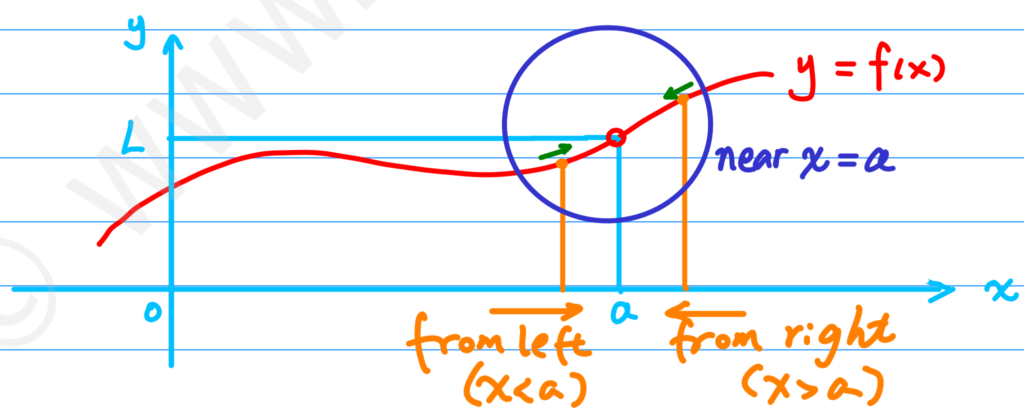
Suppose $f(x)$ is defined near the number $x = a$.

Note: $f(x)$ may or may not defined at $x = a$ itself, but is defined when x is near / close to a.

E.g.



If we can make $f(x)$ "arbitrarily" close to the number L by taking x "sufficiently" close to a



Note:
- x approaches a as close as wanted
- but $\boxed{x \neq a}$
- x approaches a $\boxed{\text{from both sides}}$

(i.e. from left : $x < a$, $x \to a$
from right : $x > a$, $x \to a$)

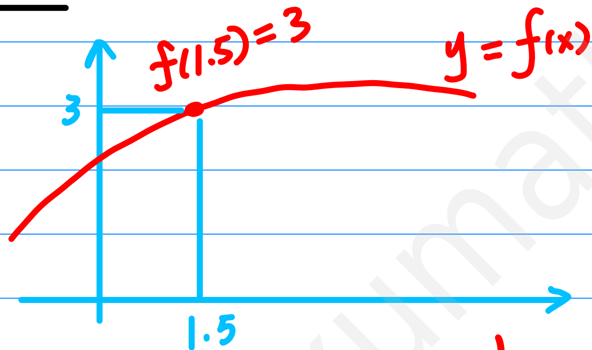
Then we say the limit of $f(x)$, as x approaches a, equals L, and write the statement as:

$$\lim_{x \to a} f(x) = L$$

(or: $f(x) \to L$ as $x \to a$)

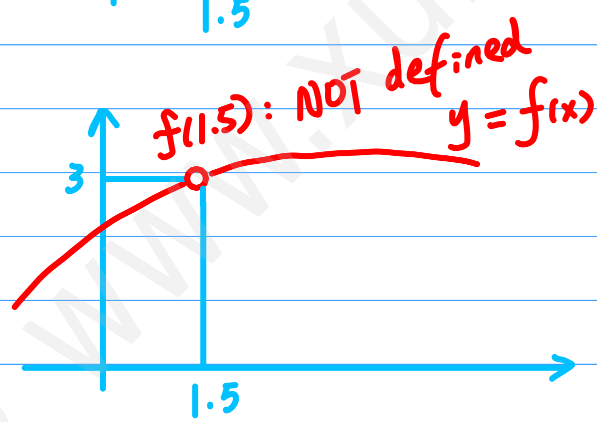
<u>Examples</u>

(1)



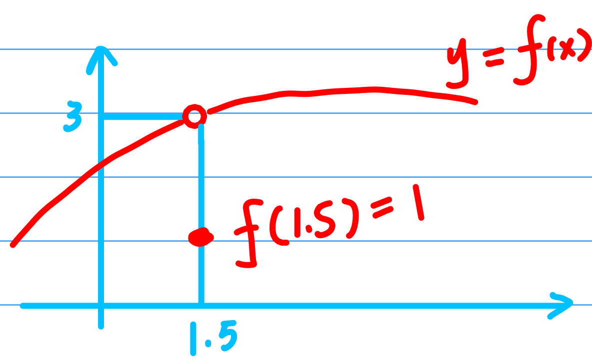
$$\lim_{x \to 1.5} f(x) =$$

(2)



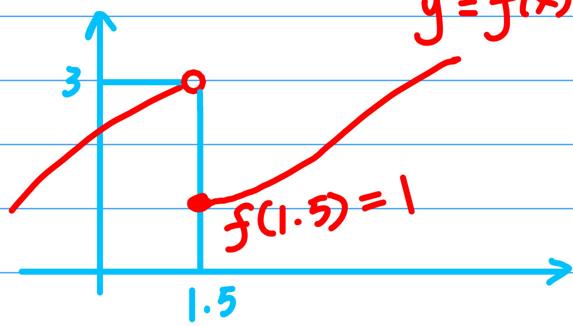
$$\lim_{x \to 1.5} f(x) =$$

(3)



$$\lim_{x \to 1.5} f(x) =$$

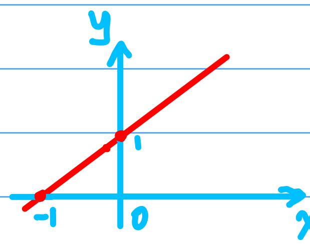
(4)



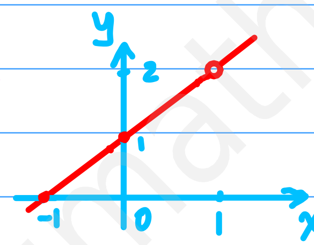
$$\lim_{x \to 1.5} f(x) =$$

poll Which one is the graph of $f(x) = \dfrac{x^2-1}{x-1}$?

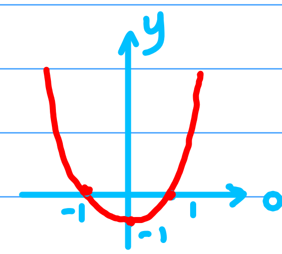
A.



B.



C.



(5) $$\lim_{x \to 1} \frac{x^2-1}{x-1} = ?$$

- $f(x) = \dfrac{x^2-1}{x-1}$: NOT defined at $x = 1$
  BUT defined near $x = 1$

- $x \to 1$: x approach 1, but $\underline{x \neq 1}$

So if $x \neq 1$, then $x - 1 \neq 0$ and

$$f(x) = \frac{x^2-1}{x-1} = \frac{(x+1)(x-1)}{(x-1)} = x+1$$

$$\lim_{x \to 1} \frac{x^2-1}{x-1} \overset{x \neq 1}{=} \lim_{x \to 1} (x+1) = 2$$

Existence of a Limit

E.g. $\lim_{x \to 0} \frac{\sin x}{x} = ?$

As $x \to 0$: numerator $\sin x \to 0$
denominator $x \to 0$
So $\frac{\sin x}{x} \to \frac{0}{0}$?

Note: $f(x) = \frac{\sin x}{x}$ is an even function

$$f(-x) = \frac{\sin(-x)}{(-x)} = \frac{-\sin x}{-x} = \frac{\sin x}{x}$$
$$= f(x)$$

x	$f(x) = (\sin x)/x$
± 0.1	calculartor: x in radians
± 0.01	
± 0.001	
± 0.0001	

Guess: $\lim_{x \to 0} \frac{\sin x}{x} = 1$

Graphs: $y = \sin x,\ x,\ \frac{\sin x}{x}$
(desmos.com/calculator)

Fact: $\lim_{x \to 0} \frac{\sin x}{x} = 1$

As $x \to 0$, $\sin x \to 0$ with the same speed as $x \to 0$

E.g. $\lim_{x \to 0} \sin\left(\frac{1}{x}\right) = ?$

Note: $f(x) = \sin\left(\frac{1}{x}\right)$ is NOT defined only at $x = 0$.

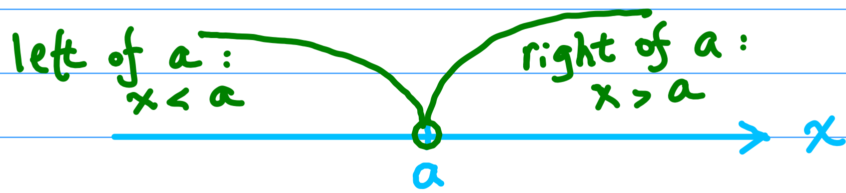
Graph: desmos.com

Fact: $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$ $\underbrace{\text{DNE}}_{\text{does not exist}}$

poll $\lim_{x \to 0} x \cdot \sin\left(\frac{1}{x}\right) = ?$

A. 0 B. 1 C. DNE

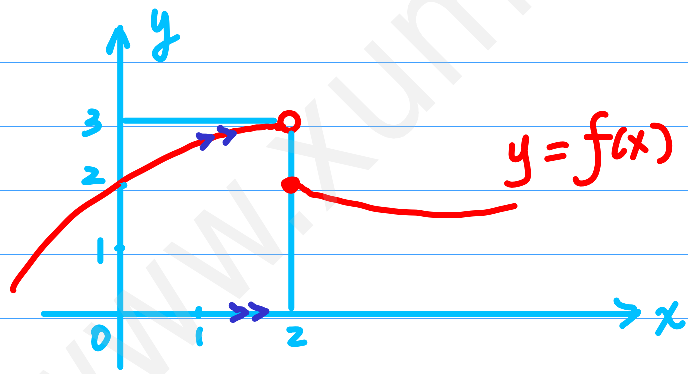
One-Sided Limits



- If x approaches a from the left only we write $x \to a^-$ $(- : x < a)$
- If x approaches a from the right only we write $x \to a^+$ $(+ : x > a)$

The left-hand limit $\lim\limits_{x \to a^-} f(x)$

E.g.

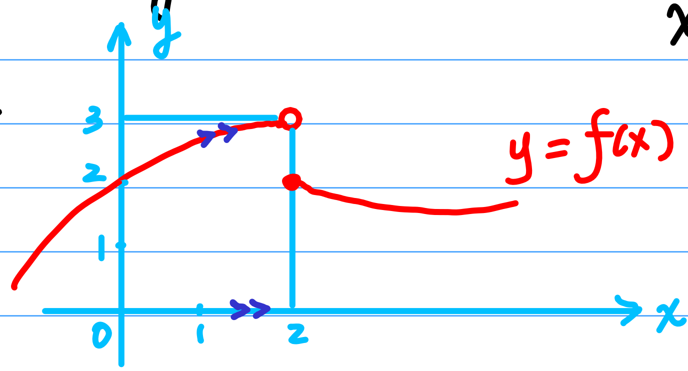


$\lim\limits_{x \to 2^-} f(x) =$

Note: Don't care what's going on at a or at the right of a.

The right-hand limit $\lim\limits_{x \to a^+} f(x)$

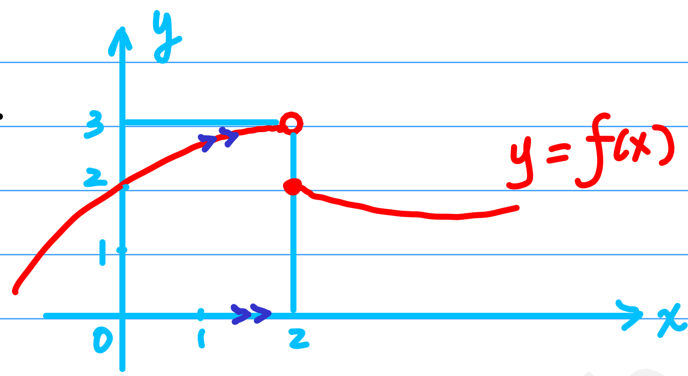
E.g.



$\lim\limits_{x \to 2^+} f(x) =$

* the two-sided limit $\lim_{x \to a} f(x)$ and the one-sided limits

E.g.



$$\lim_{x \to 2^-} f(x) = 3$$

$$\lim_{x \to 2^+} f(x) = 2$$

$$\lim_{x \to 2} f(x) \text{ DNE}$$

$$\boxed{\lim_{x \to a} f(x) = L \overset{\text{if and}}{\underset{\text{only if}}{\Longleftrightarrow}} \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L}$$

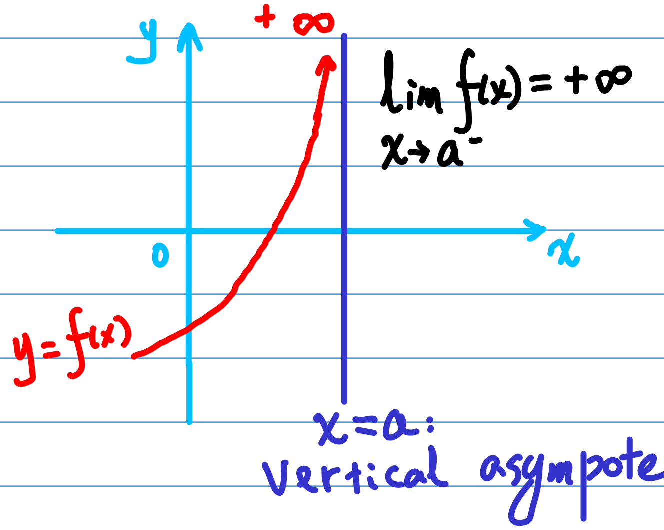
poll The Heaviside (or step) function is a piecewise function

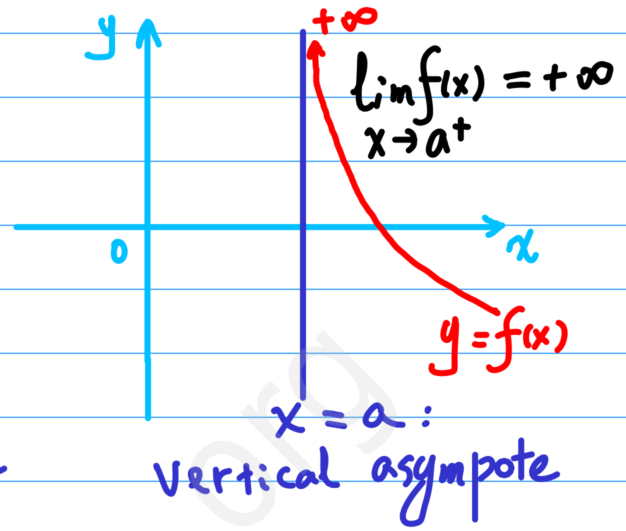
$$H(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}$$

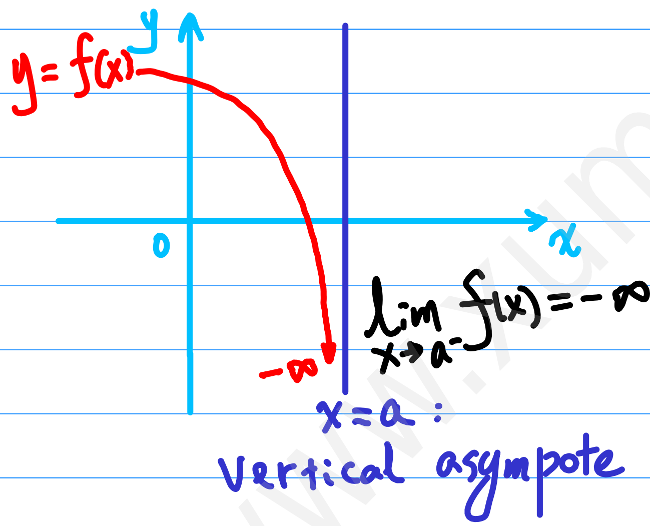
$$\lim_{x \to 0^+} H(x) = ?$$

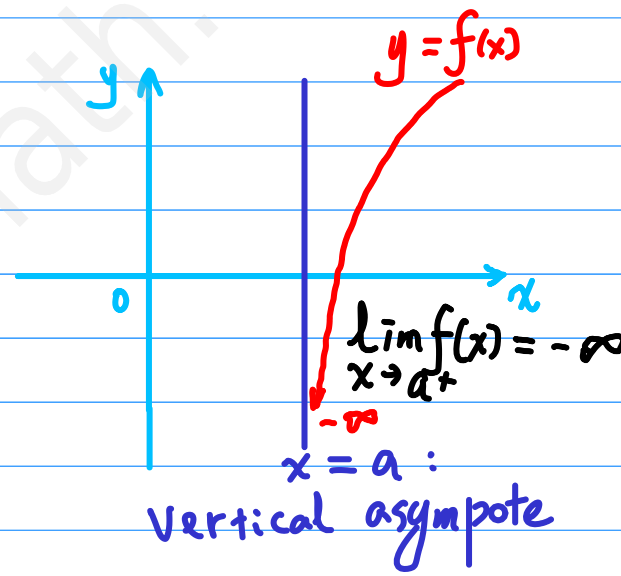
A. 0 B. 1 C. DNE

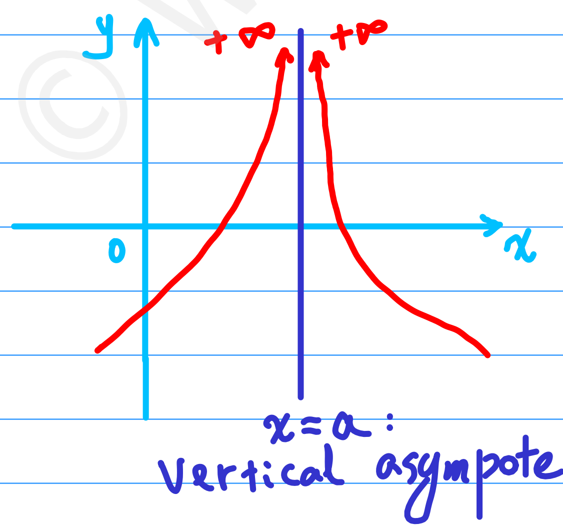
Infinite Limits



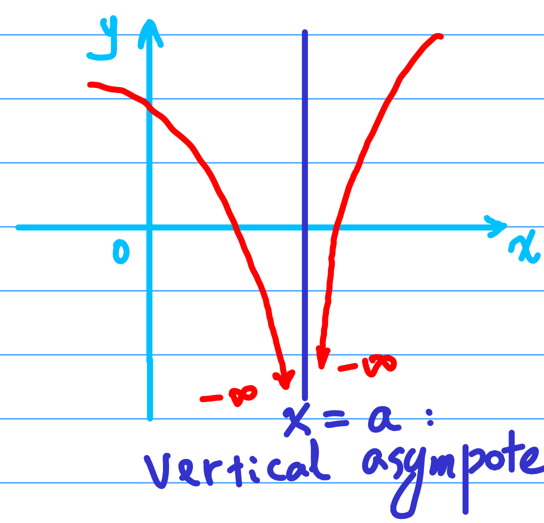








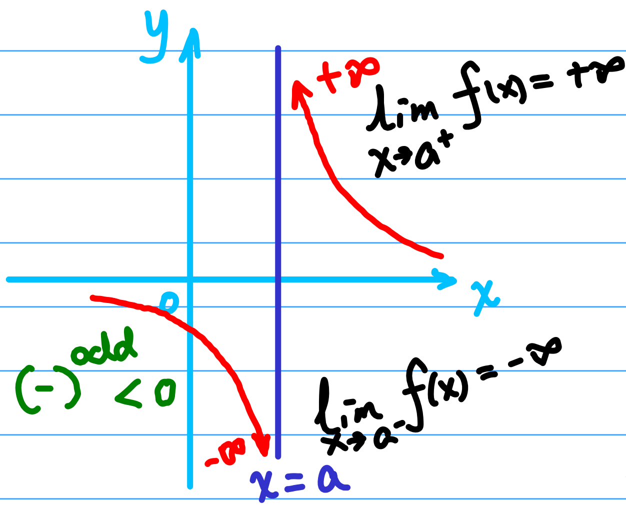
$$\lim_{x \to a} f(x) = +\infty$$

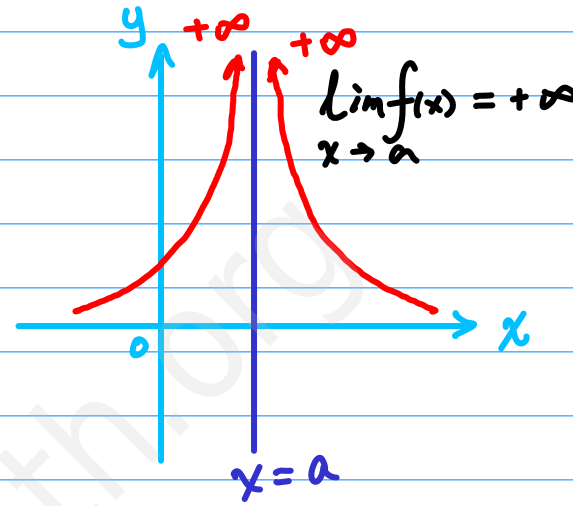


$$\lim_{x \to a} f(x) = -\infty$$

* Graphs of $f(x) = \dfrac{1}{\boxed{(x-a)^n}}$

$(x-a)^n \to 0$ as $x \to a$





- if n is an odd positive integer
  eg. $f(x) = \dfrac{1}{x-a}$

- if n is an even positive integer
  eg. $f(x) = \dfrac{1}{(x-a)^2}$

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$$\lim_{v\to c^-} \frac{m}{\sqrt{1-\frac{v^2}{c^2}}} = ?$$

A. 0 B. $+\infty$ C. $-\infty$