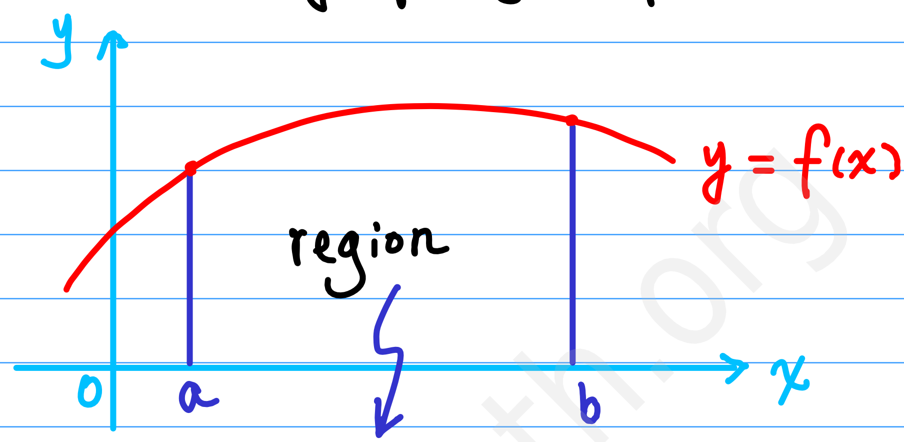
# A Preview of Calculus

## The Area Problem

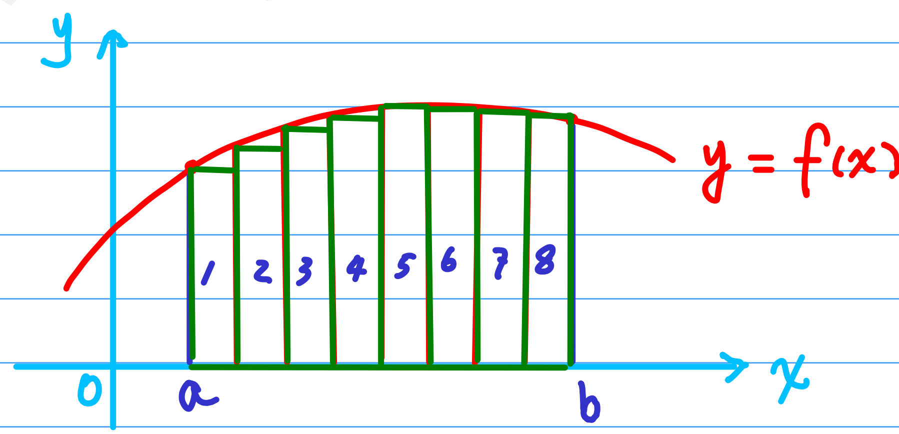
* Area under the graph of a function




under the curve $y = f(x)$ and
above the $x$-axis
between the lines $x = a$, $x = b$

(1) Divide the region into $n$ strips with same width ; Approximate each strip as a rectangle.

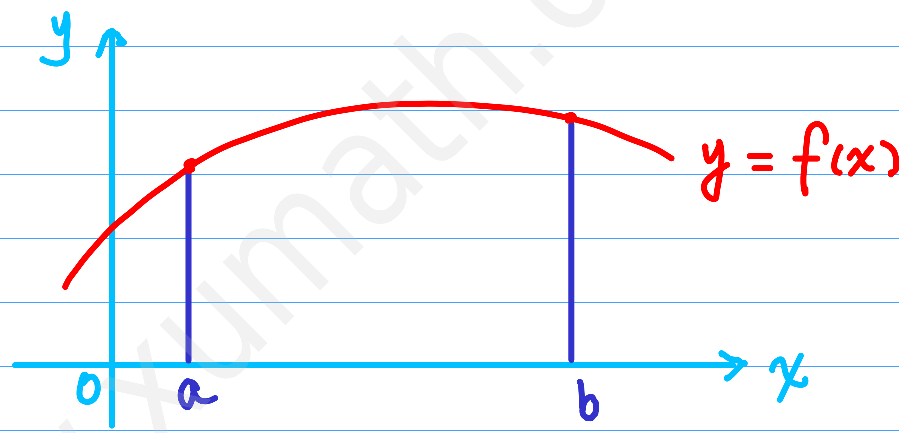
$n = 8$



(2) Approximate the area of the region by the sum of the areas of the rectangles

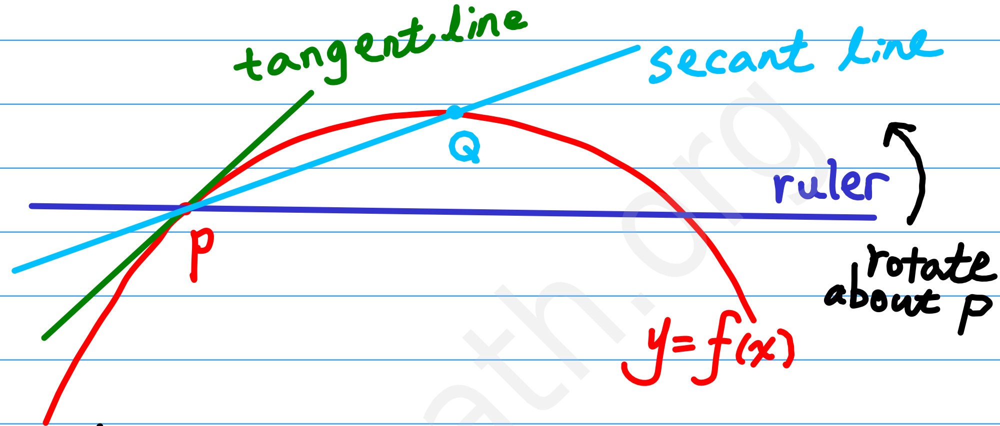
(3) The limit process : Let $n \to \infty$
The approximation approaches the area of the region.

$n \to \infty$



# The tangent/velocity Problem

* The slope of the tangent line to the curve $y = f(x)$ at a point




- The limit process:

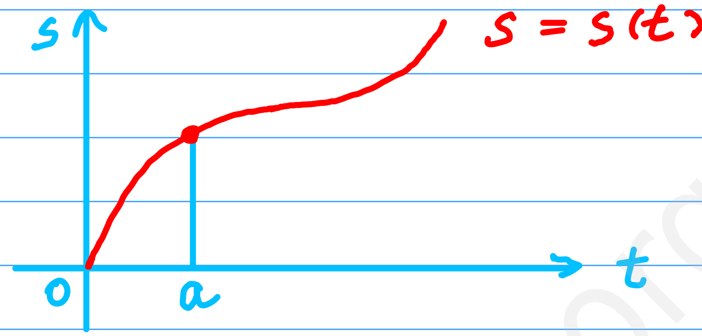
  The ruler is rotated about $P$ s.t. point $Q \to$ point $P$.

  As $Q \to P$ the secant line $PQ$ approaches the tangent line at $P$.

  slope of the tangent line $m$:

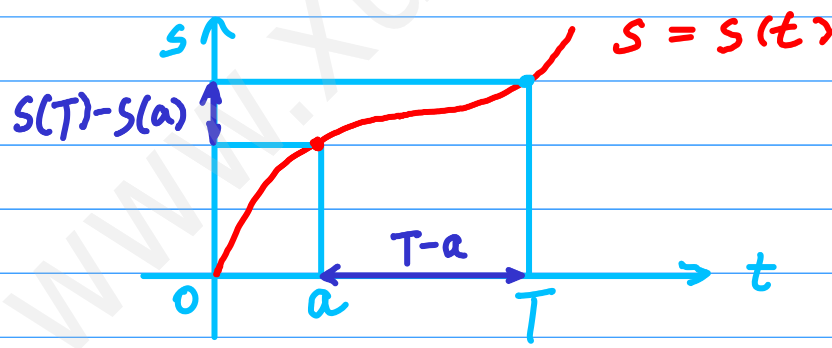
$$\underbrace{m_{PQ}}_{\text{slope of secant}} \longrightarrow m \quad \text{as } Q \to P$$

* If the displacement is not a linear function of time :




Find the instantaneous velocity at $t = a$.

(1) Find the average velocity between the instant $a$ and another instant $T\ (\neq a)$.




(2) The limiting process :

Let the other instant $T$ approach the instant $a$. i.e. $T \to a$

Then the average velocity between $a$ and $T$ approaches the instantaneous velocity at $a$.

# Calculus



- Instantaneous rate of change
  (Differentiation calculus)
- Accumulation of changing quantities
  (Integration calculus)

The limit process

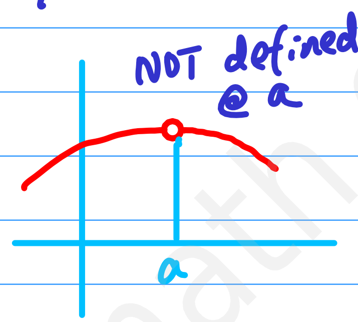
# The Limit of A Function

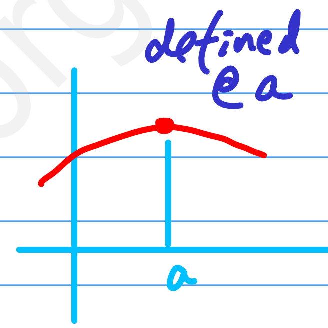
## Intuitive Definition of a Limit

Suppose $f(x)$ is defined near the number $x = a$.

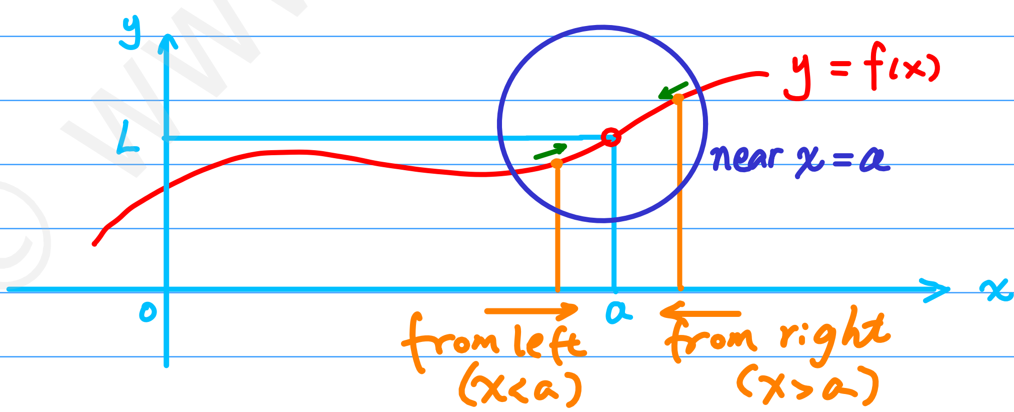
Note: $f(x)$ may or may not defined at $x = a$ itself, but is defined when $x$ is near / close to $a$.

E.g.




If we can make $f(x)$ "arbitrarily" close to the number $L$ by taking $x$ "sufficiently" close to $a$




Note:
- $x$ approaches $a$ as close as wanted
- but $\boxed{x \neq a}$
- $x$ approaches $a$ from both sides
  (i.e. from left : $x < a$, $x \to a$
  from right : $x > a$, $x \to a$ )

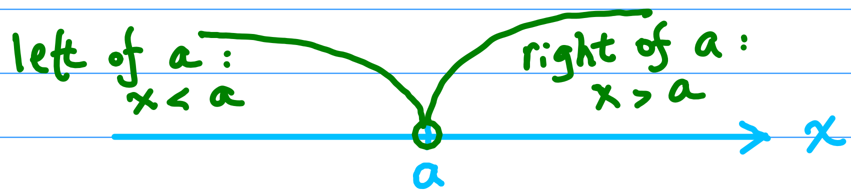
Then we say the limit of $f(x)$, as $x$ approaches $a$, equals $L$, and write the statement as:

$$\lim_{x \to a} f(x) = L$$

(or: $f(x) \to L$ as $x \to a$)

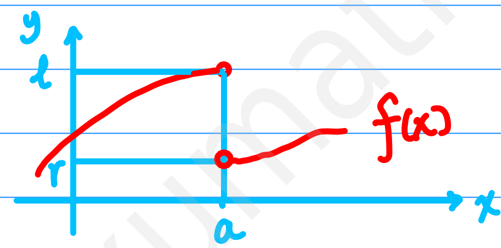
# One-Sided Limits






- If $x$ approaches $a$ from the left only, we write $x \to a^-$ $(- : x < a)$
- If $x$ approaches $a$ from the right only, we write $x \to a^+$ $(+ : x > a)$

$$\lim_{x \to a^-} f(x) = l$$




$$\lim_{x \to a^+} f(x) = r$$

The left-hand limit $\lim_{x \to a^-} f(x)$

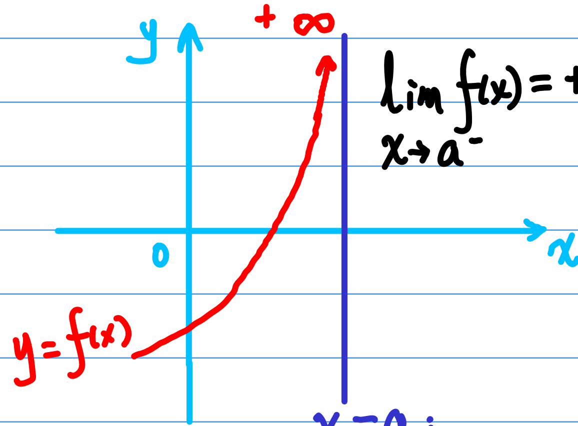
Note: Don't care what's going on at a or at the right of a.

The right-hand limit $\lim_{x \to a^+} f(x)$

Note: Don't care what's going on at a or at the left of a.

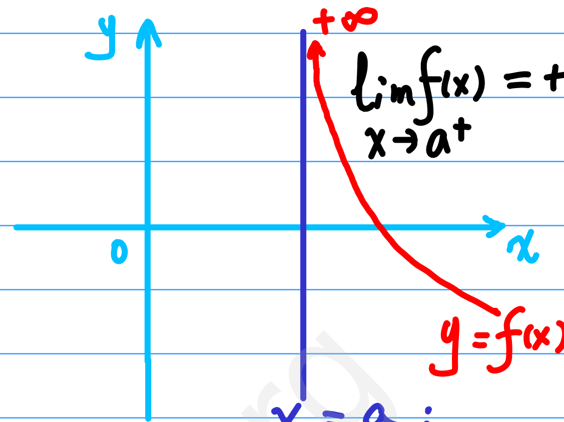
$$\boxed{\lim_{x \to a} f(x) = L \overset{\text{if and}}{\underset{\text{only if}}{\Longleftrightarrow}} \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L}$$

# Infinite Limits



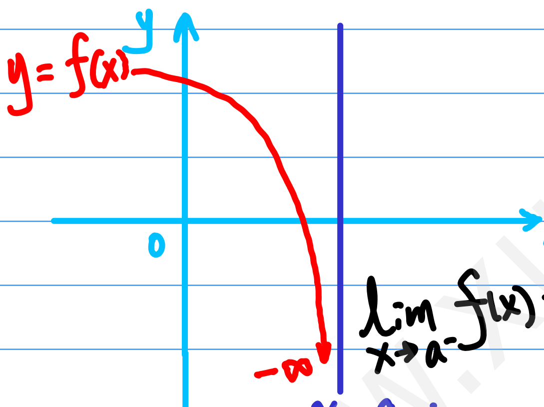

$$\lim_{x\to a^-} f(x) = +\infty$$

$x = a$: Vertical asymptote



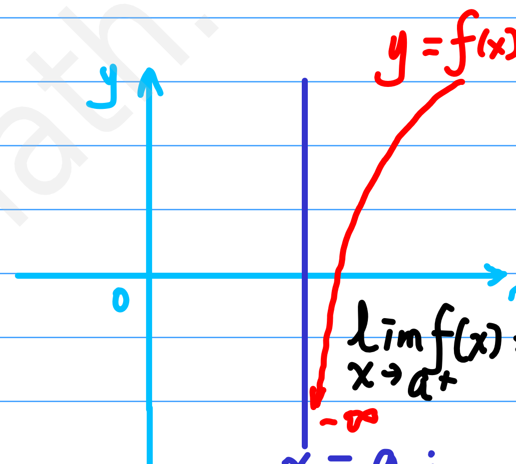

$$\lim_{x\to a^+} f(x) = +\infty$$

$x = a$: Vertical asymptote



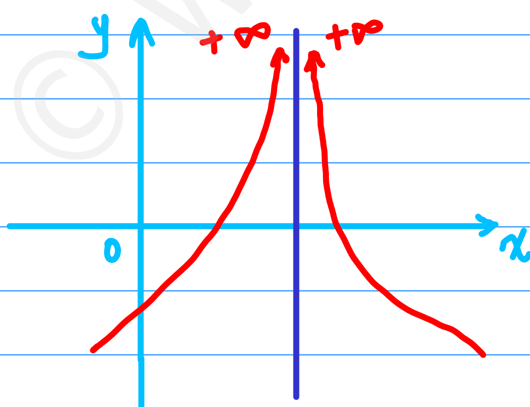

$$\lim_{x\to a^-} f(x) = -\infty$$

$x = a$: Vertical asymptote



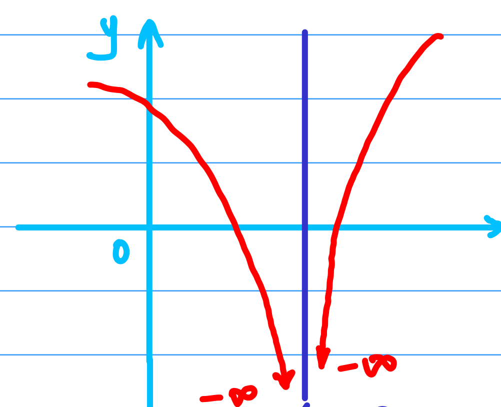

$$\lim_{x\to a^+} f(x) = -\infty$$

$x = a$: Vertical asymptote




$x = a$: Vertical asymptote

$$\lim_{x\to a} f(x) = +\infty$$

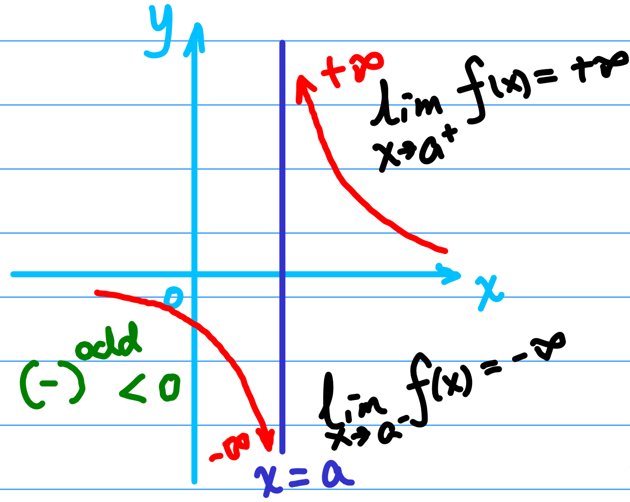


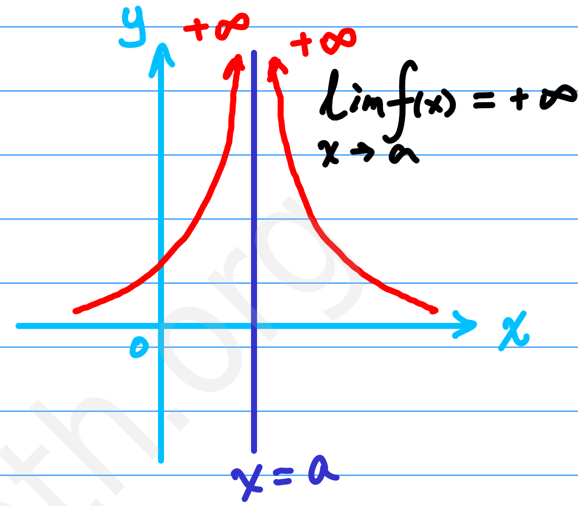

$x = a$: Vertical asymptote

$$\lim_{x\to a} f(x) = -\infty$$

* Graphs of $f(x) = \dfrac{1}{\boxed{(x-a)^n}}$

$(x-a)^n \to 0$ as $x \to a$




- if $n$ is an odd positive integer

  eg. $f(x) = \dfrac{1}{x-a}$

- if $n$ is an even positive integer

  eg. $f(x) = \dfrac{1}{(x-a)^2}$