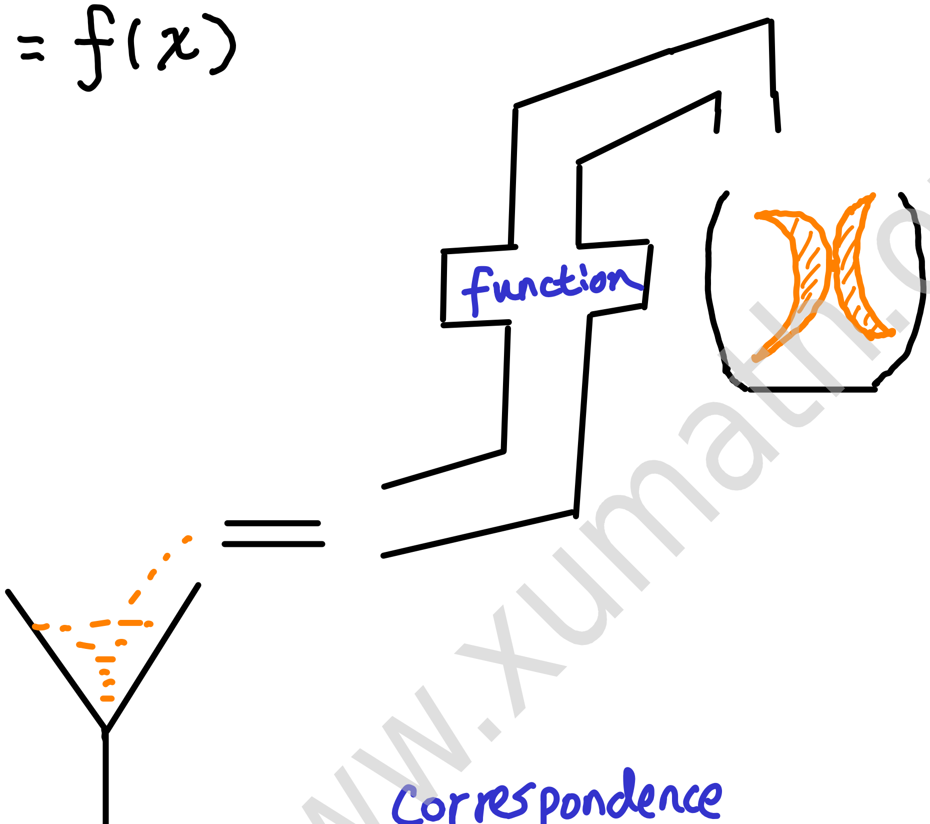
§1.2-1.5

Functions : Definition, Graphs, Operations

<u>An analogy</u> : Juice machine

$y = f(x)$



Correspondence
(mapping,
relation)

apple $\longrightarrow$ apple juice

banana $\longrightarrow$ banana juice

grape $\longrightarrow$ grape juice

One input produce $\boxed{\text{ONE}}$ output !

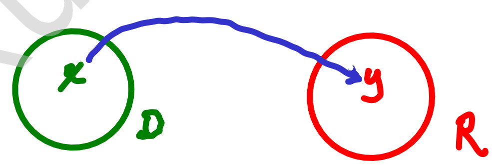
A mathematical function

* Definition

A function f from a set D to a set E is a correspondence that assigns to each element x of D $\boxed{\text{ONE}}$ element y of R.

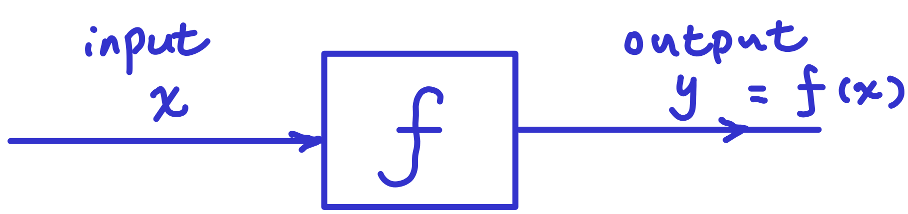
- Notations: $D \xrightarrow{f} R$

$f : D \to R$



$$\boxed{y = f(x)}$$

- Illustration:



f: name of the function,
rule to make y using x

eg. If $f(x) = x^2 - x$ then $y = x^2 - x$

When $x = 1$: $y = f(1) = 1^2 - 1 = 0$

$x = 2$: $y = f(2) = 2^2 - 2 = 2$

- Domain and Range of f:

$$\boxed{\underbrace{y}_{\text{dependent variable}} = f(\underbrace{x}_{\text{independent variable}})}$$

Domain D: Set D from which independent variable x takes its values

Range R: set R that contains all possible values y $(= f(x))$ for x in D

$$\boxed{x \in D} \xrightarrow{f} \boxed{y = f(x) \in R}$$

Note: If D is not specified, implied D is set of all possible x values for which $f(x)$ is defined.

eg. $f(x) = \frac{1}{x}$

Domain : $x \neq 0$

set of all real numbers

i.e. $D = \{x \mid x \in \mathbb{R}, x \neq 0\}$

in

Range : $y = \frac{1}{x} \Rightarrow x = \frac{1}{y}$

(any nonzero y-value can be generated by $x = \frac{1}{y}$ as the input)

So $y \neq 0$, $R = \{y \mid y \in \mathbb{R}, y \neq 0\}$

- Note: $y = f(x)$ produces $\boxed{\text{ONE}}$ y-value for each x-value

eg. (1) $y = x^2$: y is a function of x

when $x = 1$: $y = 1$ (one y-value)

$x = -1$: $y = 1$ (one y-value)

OK to produce the same y from different x.

(2) $x = y^2$, $x \geq 0$:

when $x = 1$: $y = 1$ or -1

not OK to produce different y from the same x

y is NOT a function of x

$\boxed{\text{Poll 1}}$ Does the equation $|y| = x$ define a function $y = f(x)$?

A. Yes B. No.

* Evaluating a function

eg. $f(x) = |x-1|$

$f(2) = |2-1| = 1$

$f(-2) = |-2-1| = 3$

Note: $f(x) = \begin{cases} x-1, & \text{if } x \geq 1 \\ -(x-1) = 1-x, & \text{if } x \leq 1 \end{cases}$

is a piecewise-defined function

$f(2) = ?$: $2 \geq 1 \Rightarrow f(2) = 2-1 = 1$

$f(-2) = ?$: $-2 \leq 1 \Rightarrow f(-2) = 1-(-2) = 3$

- $f(x)$ = Expression of x

a place holder

eg. $f(x) = \dfrac{1}{x-1}$: $f(\square) = \dfrac{1}{\square - 1}$

$$f(2+1) = \frac{1}{(2+1)-1} = \frac{1}{2}$$

$$f(a^2-a+1) = \frac{1}{(a^2-a+1)-1} = \frac{1}{a^2-a}$$

$\boxed{\text{poll2}}$ $f(x) = \begin{cases} x^2-1, & \text{if } x<1 \\ x-1, & \text{if } x \geq 1 \end{cases}$; $f(a^2+1) = ?$

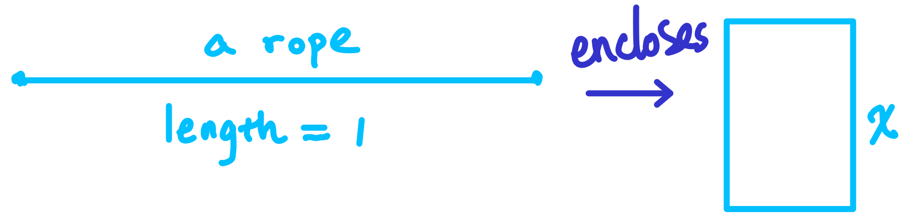
A. a^4+2a^2 B. a^2 C. a^2+1

* Modeling using a function

eg. $y = \underset{\text{slope}}{m}\, x + \underset{y\text{-intercept}}{b}$:

a linear function which models linear relation

eg.



To enclose a rectangle with one side of length x.

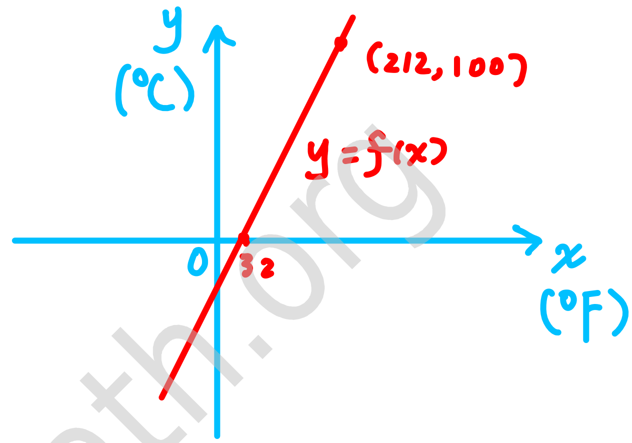
The area $y = \underbrace{x \cdot \left(\frac{1-2x}{2}\right)}_{f(x):\ \text{function of } x}$

Graphs of Functions

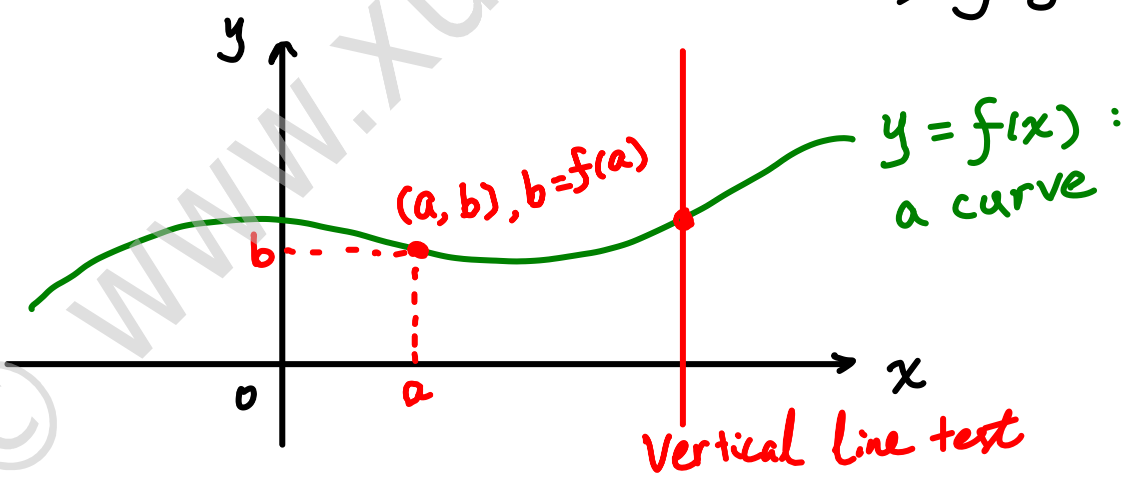
eg. x : temperature in °F
y : temperature in °C } $y = f(x)$

$$f(x) = \frac{5}{9}(x - 32)$$

x (°F)	y (°C)
32	0
212	100
⋮	⋮

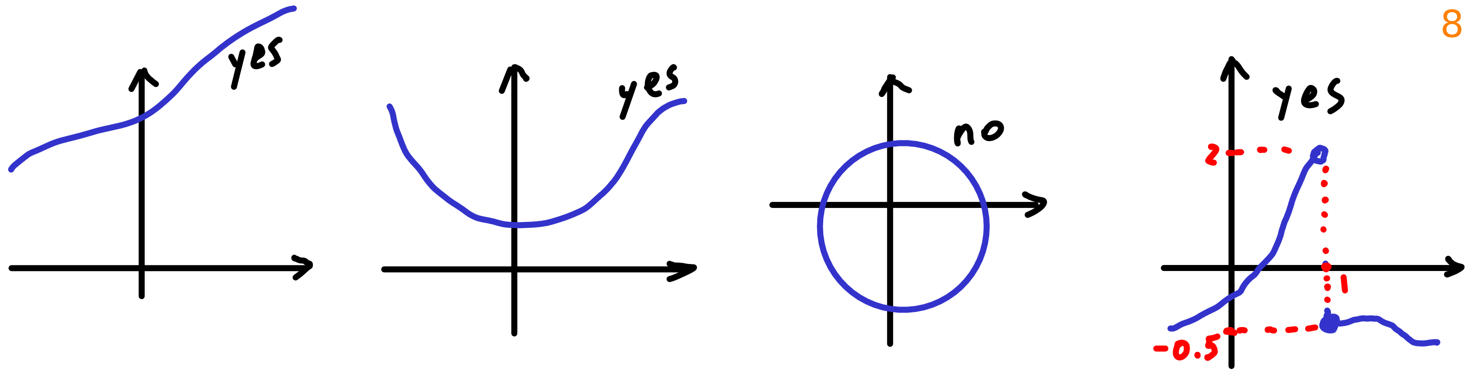


* Graph of $y = f(x)$: all points (x, y), where $x \in D$, $y = f(x)$



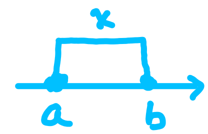
$\boxed{\text{Vertical Line test}}$:

A curve in the coordinate plane is the graph of a function iff (if and only if) no vertical line intersects the curve more than once.

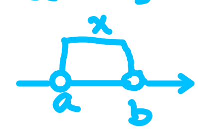


* Increasing and decreasing f

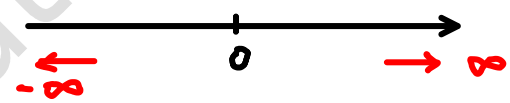
$x \in [a, b]$: $a \le x \le b$,
closed interval

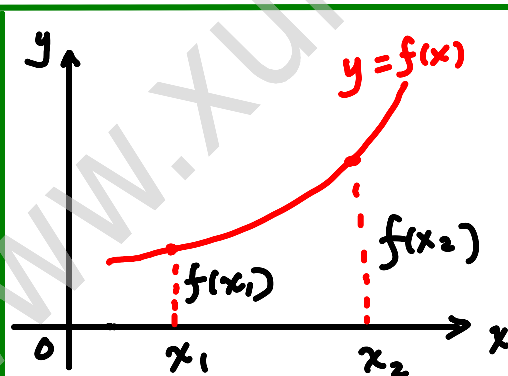
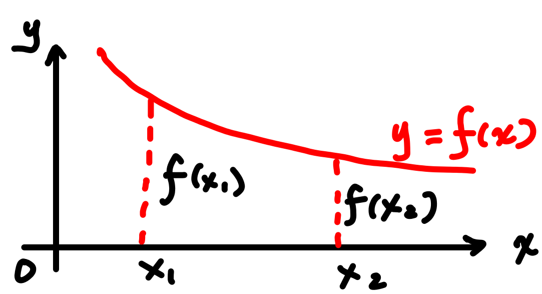
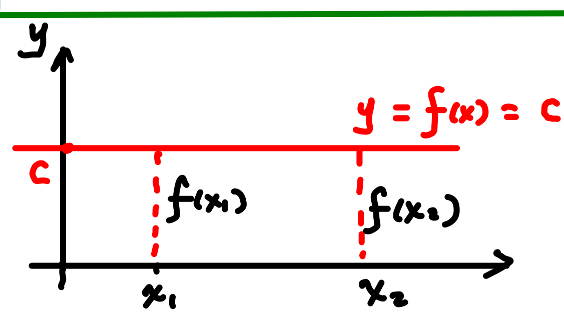


$x \in (a, b)$: $a < x < b$,
open interval



∞ & $-\infty$:
infinity (or: $+\infty$)

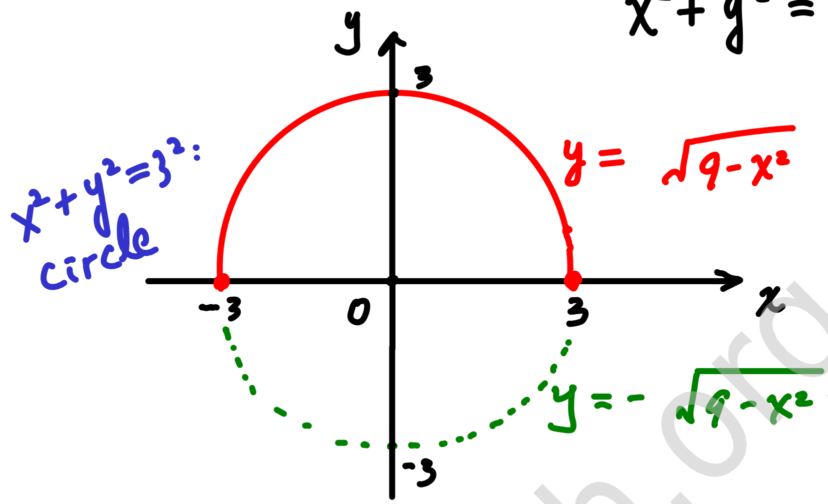


• f is increasing on [a, b]		Whenever $x_2 > x_1$, $f(x_2) > f(x_2)$
• f is decreasing on [a, b]		Whenever $x_2 > x_1$, $f(x_2) < f(x_1)$
• f is constant on [a, b]		For any x_1 and x_2, $f(x_1) = f(x_2)$

eg. $f(x) = \sqrt{9-x^2}\,, \quad -3 \le x \le 3$

$y = \sqrt{9-x^2} \geq 0 : \qquad y^2 = 9 - x^2$

$x^2 + y^2 = 3^2 \,, \; y \geq 0$



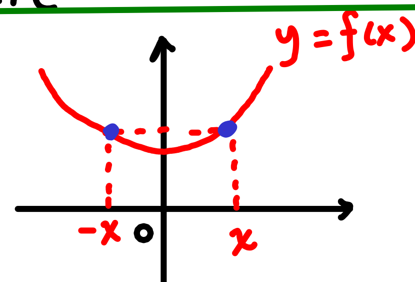
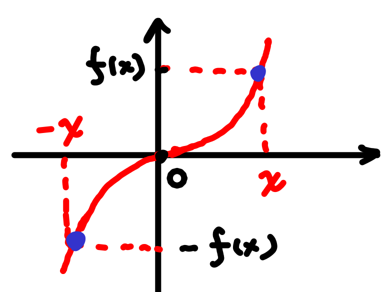
Domain : $[-3, 3]$ $(-3 \le x \le 3)$

Range : $[0, 3]$ $(0 \le y \le 3)$

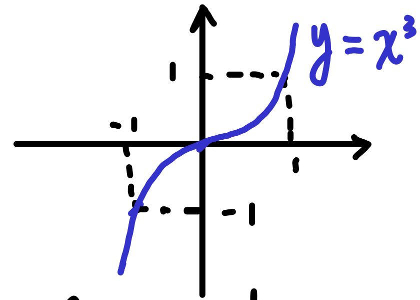
Increasing : on $[-3, 0]$

Decreasing : on $[0, 3]$

* Odd and even

• f is an even function	 symmetric about the y-axis	$f(-x) = f(x)$, $x \in D$
• f is an odd function	 symmetric about the origin	$f(-x) = -f(x)$, $x \in D$

eg. (1) $f(x) = x^3$

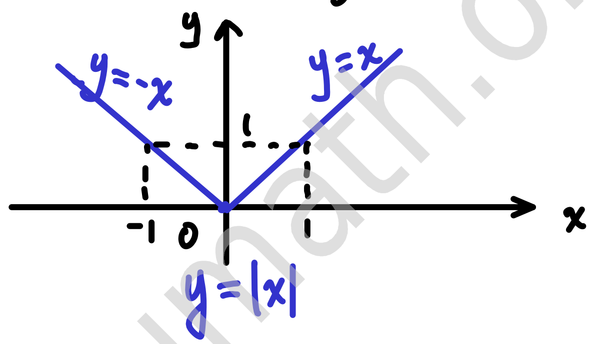


$f(x) = x^3$

$f(-x) = (-x)^3 = -x^3$

So $f(-x) = -f(x)$, f is odd

(2) $f(x) = |x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x \leq 0 \end{cases}$



$\left.\begin{array}{l} f(x) = |x| \\ f(-x) = |-x| = |x| \end{array}\right\}$ $f(-x) = f(x)$, f is even

(3) $f(x) = \underline{x^2 + x}$; $-f(x) = \underline{-x^2 - x}$

$f(-x) = (-x)^2 + (-x) = \underline{x^2 - x}$

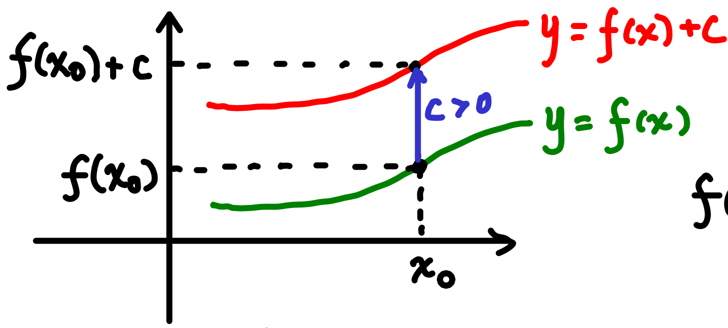
$\left.\begin{array}{l} f(-x) \neq f(x) \\ f(-x) \neq -f(x) \end{array}\right\}$: f is neither even nor odd

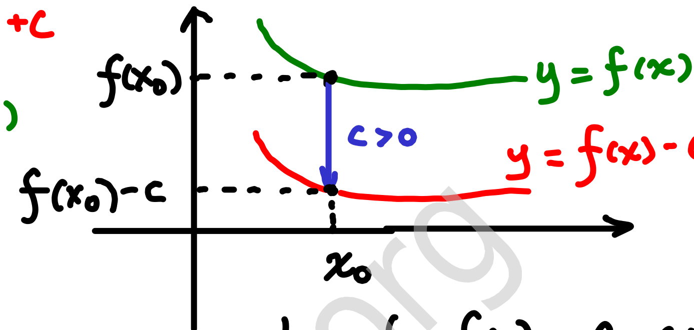
$\boxed{\text{poll 3}}$ If $f(x)$ is an odd function, and $(5, -3)$ is a point on its graph, then which point is also on its graph?

A. $(5, 3)$ B. $(-5, -3)$ C. $(-5, 3)$

Transformations of f

* Vertically shifting the graph of f

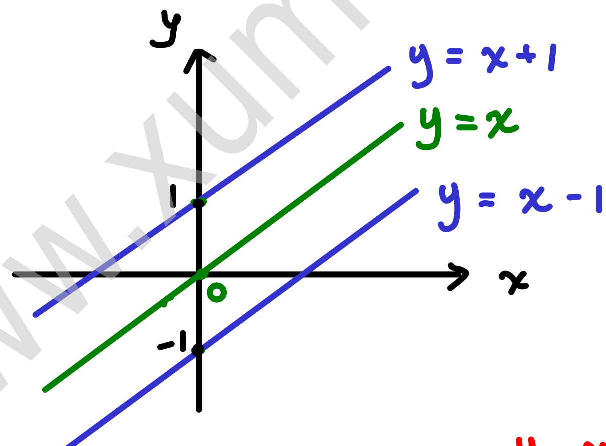




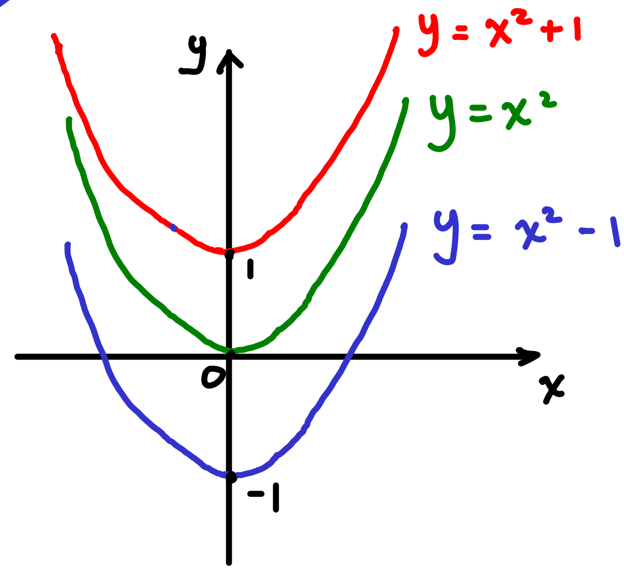
graph of $f(x)\underline{+c}$, $c>0$:
graph of $f(x)$ shifted upward by distance c

graph of $f(x)\underline{-c}$, $c>0$:
graph of $f(x)$ shifted downward by distance c

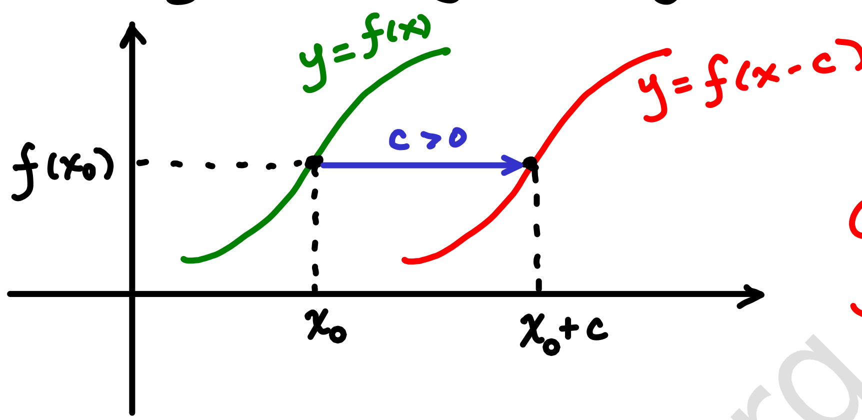
eg. (1)



(2)

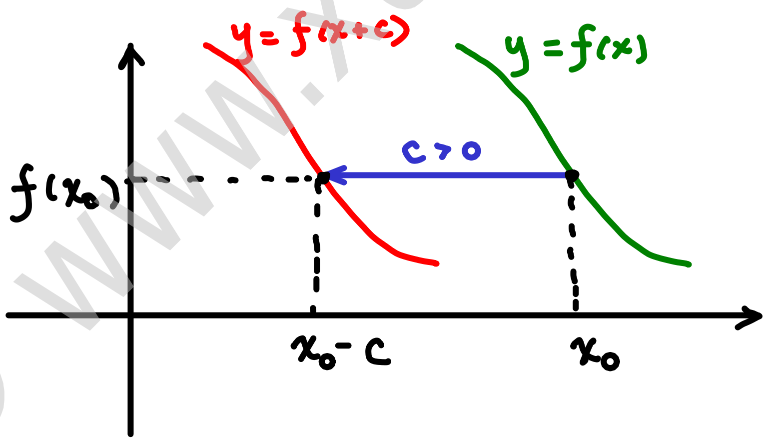


* Horizontally shifting the graph of f



$y=f(x-c)$
@ x_0+c
$y=f(x_0+c-c)$
$=f(x_0)$

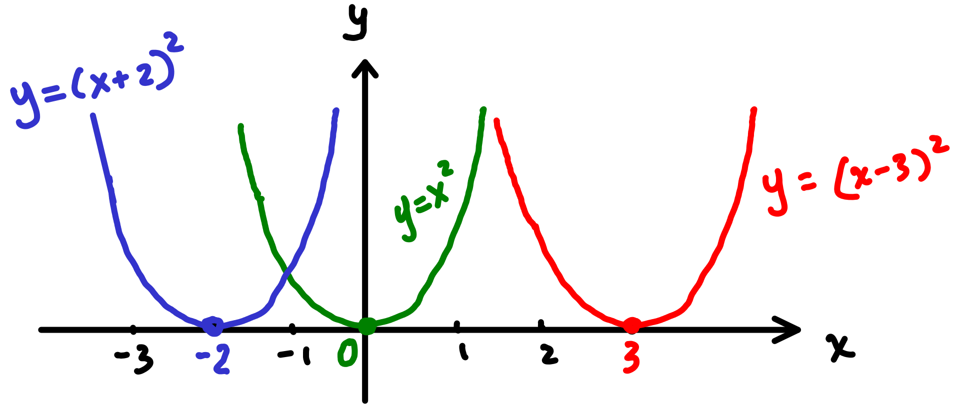
graph of $f(x-c)$, $c>0$:
graph of $f(x)$ shifted horizontally to the right by distance c



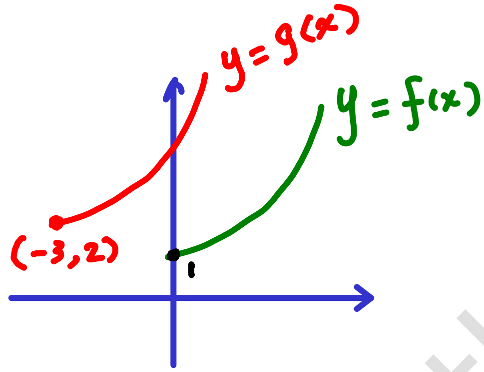
$y=f(x+c)$
@ x_0-c
$y=f(x_0-c+c)$
$=f(x_0)$

graph of $f(x+c)$, $c>0$:
graph of $f(x)$ shifted horizontall to the left by the distance c

eg.



$\boxed{\text{poll4}}$

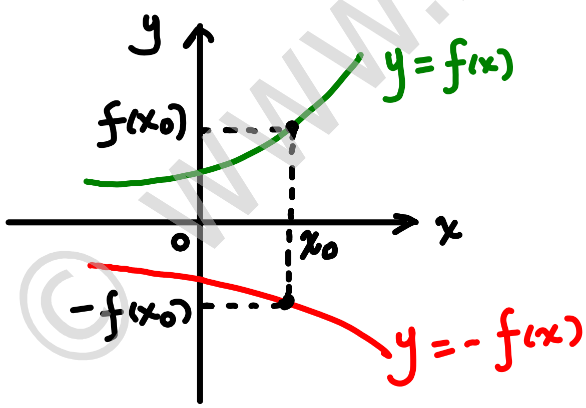


The graph of $y = g(x)$ is obtained by shifting the graph of $y = f(x)$ both vertically and horizontally.

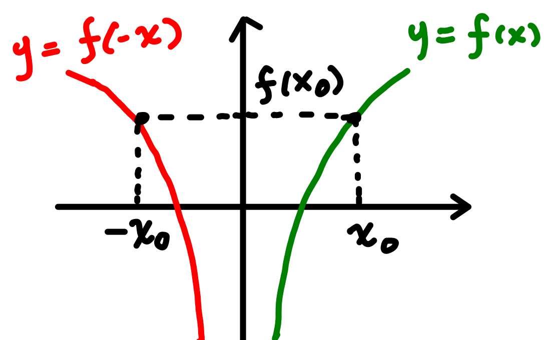
$g(x) = ?$

A. $f(x+3)+1$ B. $f(x-3)+1$ C. $f(x+3)+2$

* Reflecting the graph of f



graph of $y=-f(x)$: reflection of graph of $f(x)$ in/about the x-axis



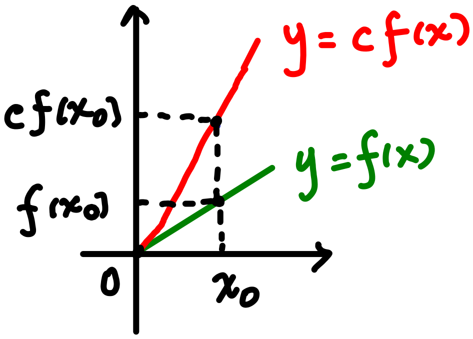
$y = f(-x)$ @ $-x_0$:

$y = f(- -x_0) = f(x_0)$

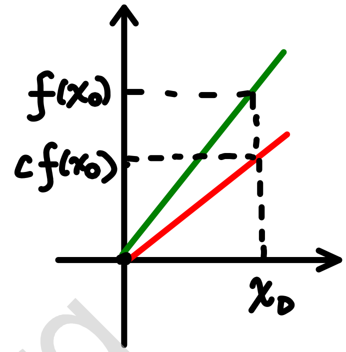
graph of $y = f(-x)$: reflection of graph of $f(x)$ in/about the y-axis

* Vertically stretching or compressing

$c > 1$: stretching



$c < 1$: compressing

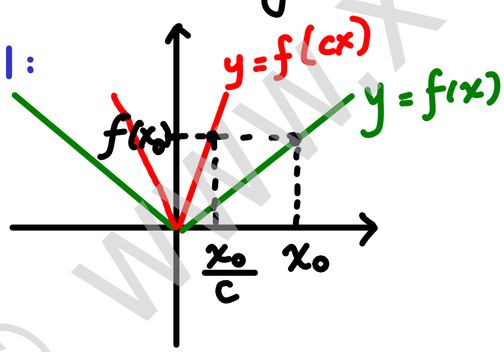


graph of $y = cf(x)$

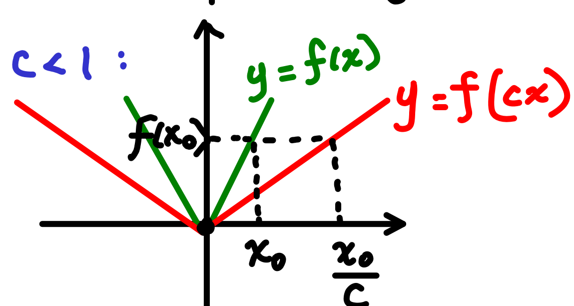
$c > 1$:	$c < 1$:
graph of f stretched vertically by a factor c	graph of f compressed vertically by a factor c

* Horizontally stretching or compressing

$c > 1$:



$c < 1$:

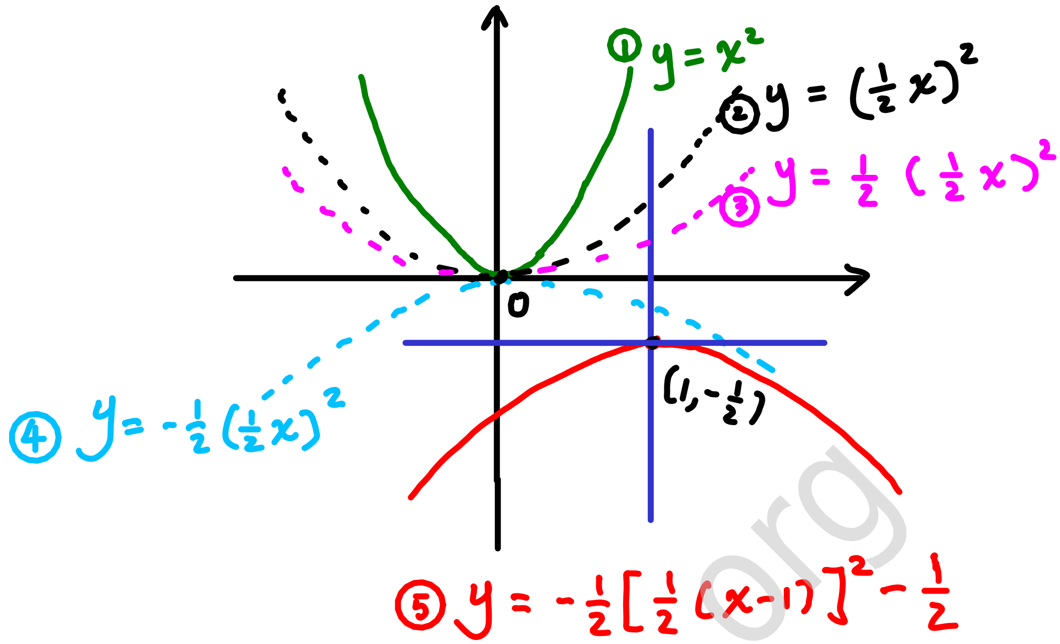


$y = f(cx)$ @ $\frac{x_0}{c}$: $y = f\left(c \frac{x_0}{c}\right) = f(x_0)$

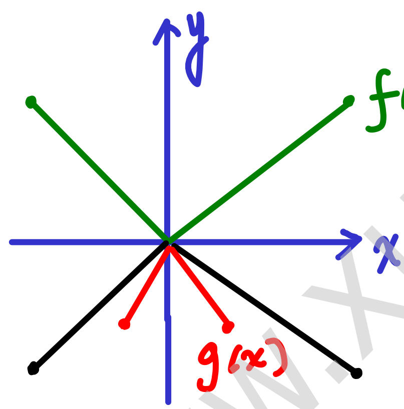
graph of $y = f(cx)$:

$c > 1$:	$0 < c < 1$:
graph of f compressed horizontally by a factor c	graph of f stretched horizontally by a factor of c

eg.



$\boxed{\text{poll 5}}$

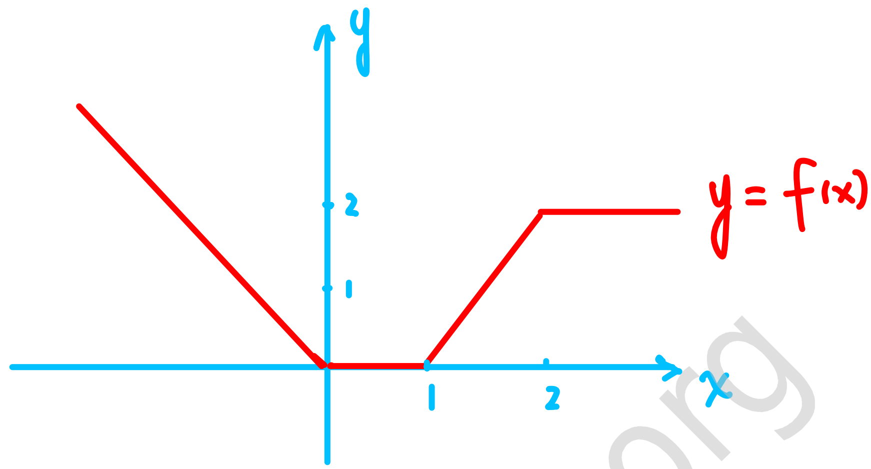


graph of g(x) is obtained from graph of f(x) by reflection and compression

Which one can be g(x) ?

A. $-2f(\frac{x}{3})$ B. $-\frac{1}{2}f(3x)$ C. $-2f(3x)$

e.g.



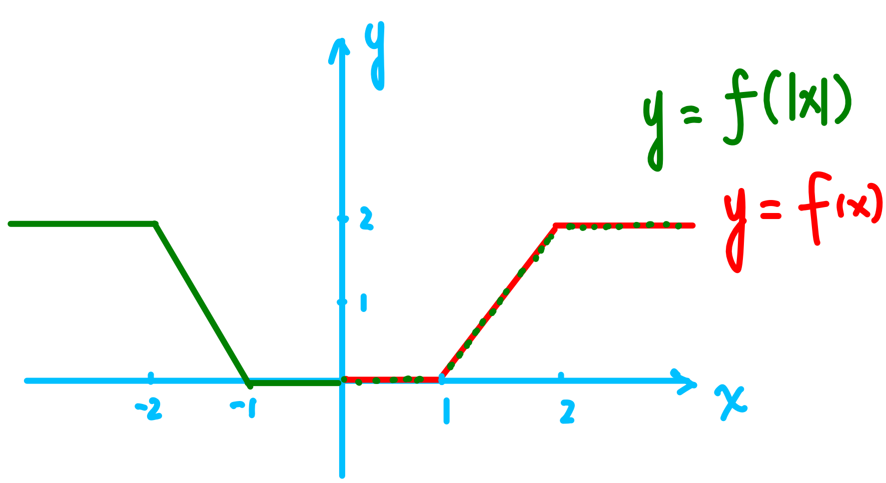
Sketch the graph of $y = f(|x|)$

[Soln:] $y = f(|x|)$ is an even function

as $f(|-x|) = f(|x|)$

Its graph is symmetric about the y-axis.

When $x \geq 0$: $|x| = x$, $f(|x|) = f(x)$

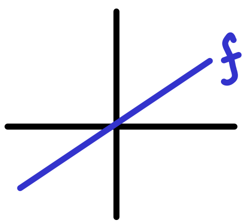


Combinations of functions

* equal functions

functions f and g are equal if they have the same domain D and $f(x) = g(x)$ for each $x \in D$.

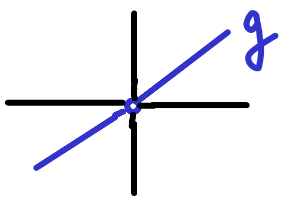
eg. $f(x) = x$, $g(x) = \frac{x^2}{x}$



Domain of f : $\mathbb{R}$

Domain of g : $x \neq 0$

$(-\infty, 0) \cup (0, \infty)$



$f \neq g$ even though For $x \neq 0$,

$g(x) = \frac{x^2}{x} = x = f(x)$

* sum, difference, product, quotient

For all x in the intersection of the domains D_1 and D_2 of f and g,

i.e. $x \in D_1 \cap D_2$

- sum : $(f+g)x = f(x) + g(x)$
- difference : $(f-g)x = f(x) - g(x)$
- product : $(fg)x = f(x) \cdot g(x)$
- quotient : $\left(\frac{f}{g}\right)x = \frac{f(x)}{g(x)}$, $g(x) \neq 0$

* Composition of functions

$$x \longrightarrow \boxed{g} \xrightarrow{g(x)} \boxed{f} \longrightarrow f(g(x))$$

eg. $g(x) = x^2$, $\quad f(x) = \sqrt{x+21}$

- $f(g(2)) = ?$

  $g(2) = 2^2 = 4$

  $f(g(1)) = f(1) = \sqrt{4+21} = 5$

  $$\xrightarrow{2} \boxed{g} \xrightarrow{4} \boxed{f} \xrightarrow{5}$$

- $f(g(x)) = ?$

  $f(x) = \sqrt{x+21}$

  $f(\underline{g(x)}) = \sqrt{\underline{g(x)} + 21} = \sqrt{x^2+21}$

- $g(f(x)) = ?$

  $g(x) = x^2$

  $g(\underline{f}(x)) = [\underline{f(x)}]^2 = [\sqrt{x+21}]^2 = x+21$

Note: $\quad f(g(x)) \neq g(f(x))$ here.

Definition:

The composition of the function f with the function g is:

$$\boxed{\underbrace{(f \circ g)}_{\text{a composite function}}(x) = f(g(x))}$$

The domain of $f \circ g$ is all the x in the domain of g such that $g(x)$ is in the domain of f.

$$\xrightarrow{x} \boxed{g} \xrightarrow{g(x)} \boxed{f} \longrightarrow \underline{\underline{f(g(x))}}$$

x: from domain of g

$\underline{g(x)}$: in the domain of f (and in the range of g)

eg. $f(x) = x^2 - 16$, $g(x) = \sqrt{x}$

(1) Find $f \circ g$ and its domain

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= [g(x)]^2 - 16 \\ &= [\sqrt{x}]^2 - 16 = x - 16\end{aligned}$$

domain of g : $[0, \infty)$

range of g : $[0, \infty)$

is inside

domain of f : $(-\infty, \infty)$

domain of $f \circ g$: $[0, \infty)$

NOT $(-\infty, \infty)$ by $f \circ g\,(x) = x - 16$

(2) Find $g \circ f$ and its domain.

$g(x) = \sqrt{x}$, $f(x) = x^2 - 16$

$$(g \circ f)\, x = g(f(x)) = \sqrt{f(x)}$$
$$= \sqrt{x^2 - 16}$$

domain of f : $(-\infty, \infty)$

domain of g : $[0, \infty)$

So need to find x in $(-\infty, \infty)$, the domain of f, such that

$f(x) = x^2 - 16 \geqslant 0$:

$x^2 \geqslant 16 \Rightarrow x \geqslant 4$ or $x \leqslant -4$

i.e. $\underbrace{x \in [4, \infty) \cup (-\infty, -4]}_{\text{in the domain of } f,\ (-\infty, \infty)}$

union

domain of $g \circ f$: $[4, \infty) \cup (-\infty, -4]$

eg. Express $y = (2x+5)^8$ as a composition of two functions.

$y = (\boxed{})^8$: $f(x) = x^8$

$g(x) = 2x+5$

outer function

So $y = (2x+5)^8 = f(\underbrace{g(x)}_{\text{inner function}}) = (f \circ g)x$

$\boxed{\text{poll6}}$ $f(x) = x^2 - 16$, $g(x) = \sqrt{x}$.

$(f \circ g)(16) = ?$

A. 240 B. -12 C. 0

eg. A spherical balloon is inflating such that its radius is increasing at a rate 3 cm/sec. Express the volume of the balloon as a function of time t (sec).

radius : $r = 3t$

volume : $V = \frac{4}{3}\pi r^3$

$= \frac{4}{3}\pi (3t)^3$

here $r = g(t)$, $V = f(r)$, $V = f(g(t))$