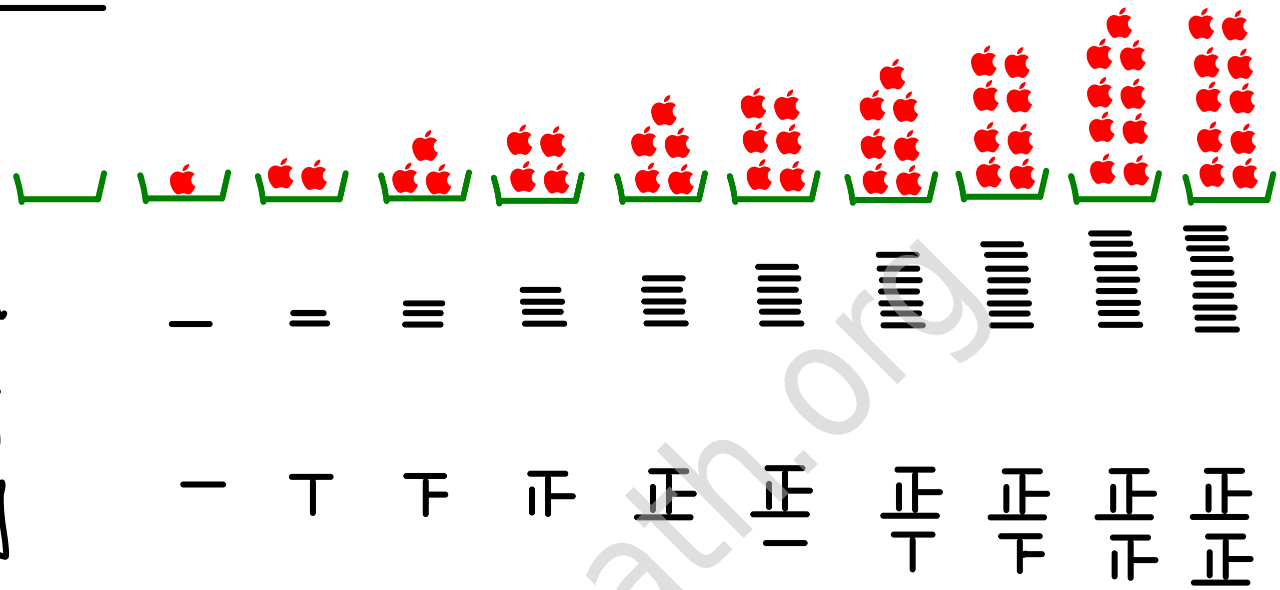
Addition, Multiplication and Subtraction

Whole numbers

apples in a basket

how many in ancient way

Chinese way



Chinese numbers	零	一	二	三	四	五	六	七	八	九	十
Roman numbers		I	II	III	IV	V	VI	VII	VIII	IX	X
Arabic numbers	0	1	2	3	4	5	6	7	8	9	10

ten digits: $\overbrace{0\ 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9}^{\text{ten digits}}$?10

natural numbers (counting numbers): $\underbrace{1\ 2\ 3\ 4\ 5\ 6\ 7 \ldots}_{\text{natural numbers (counting numbers)}}$

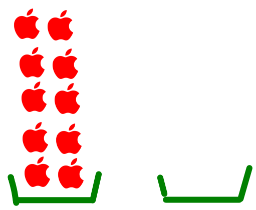
whole numbers: $\underbrace{0\ 1\ 2\ 3\ 4\ 5\ 6\ 7 \ldots}_{\text{whole numbers}}$

The nineteenth century mathematician Leopold Kronecker once proclaimed "God created the whole numbers; everything else is the work of man."

* place values (reason for carrying in addition)

Do not want to create too many symbols / digits for too many different numbers! How?

e.g.



number 10 : 1 0
place values : tens ones } 1 tens 0 ones

number 2024: 2 0 2 4
place values: thousands hundreds tens ones

$(1000 = 10 \cdot 100)$ $(100 = 10 \cdot 10)$ $(10 = 10 \cdot 1)$

$$2024 = 2 \times 1000 + 0 \times 100 + 2 \times 10 + 4 \times 1$$

* Practice

1. $35089 = 3 \times \_\_\_\_ + 5 \times \_\_\_\_ + 0 \times \_\_\_\_$
$+ 8 \times \_\_\_\_ + 9 \times \_\_\_\_$
$= 35 \times \_\_\_\_ + \_\_\_\_$

2. $5 \times 100000 + 6 \times 1000 + 3 \times 100 + 2 \times 10 = \_\_\_\_\_\_$

3. $9999 + 8888 = \_\_\_\_\_\_$

4. As we use ten digits (0, 1, 2, 3, 4, 5, 6, 7, 8, 9) and place values to represent any whole number, we use the $\underbrace{\text{deci}}_{\text{means tenth}}\text{mal}$ number system (or say base-10 system). In a digital world, people use a binary (base-2) number system in which only two digits 0 and 1 to represent a number. For example

binary number: 1 1 0 1

place values: eights (8 = 2×4) fours (4 = 2×2) twos (2 = 2×1) ones

decimal value: $1 \times 8 + 1 \times 4 + 0 \times 2 + 1 \times 1 = 13$

(1) What is the decimal value of the binary number 1011 ?

(2) What is the binary number that has the decimal value 9 ?

(3) What is the binary number that has the decimal value 21 ?
(Hint: The place value to the left of eights place is $16 = 2 \times 8$)

Addition

* Variables

  We use letters such as a, b, c, x, y, z to represent numbers. These letters are called variables.

  Let a represent a whole number. Then a may take any value as needed.

* Addition is **commutative**.

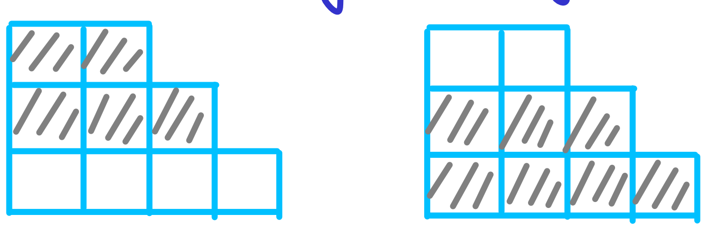
  involving changing orders

$$3+2 = 2+3$$

$$\boxed{a+b = b+a}$$

* Addition is **associative**.

  involving forming groups



$$(2+3)+4 = 2+(3+4)$$

$$\boxed{(a+b)+c = a+(b+c)}$$

* Commutative and associative properties together
When you add a lot of numbers, you can change the order of the numbers and form groups of numbers in parantheses as convenient as you like.

e.g. $472 + (219 + 28)$
$= (472 + 28) + 219 = 500 + 219 = 719$

* Adding zero.

$$\boxed{a + 0 = a}$$

Adding 0 to any number does not change the number.

* Practice

5. $76 + 397 + 24 =$ ______

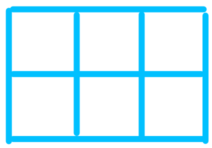
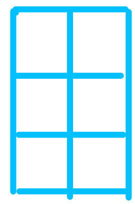
6. $(2+12+22+32) + (8+18+28+38) =$ ______

7. $1 + 2 + 3 + \underbrace{\cdots}_{\text{including all the numbers in the pattern}} + 19 + 20 =$ ______

Multiplication

* Multiplication is commutative.

e.g.

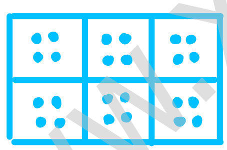
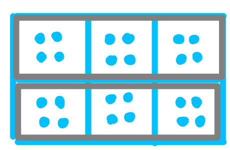



$$2 \times 3 = 3 \times 2$$

$$\boxed{\begin{aligned} a \times b &= b \times a \\ a \cdot b &= b \cdot a \\ ab &= ba \end{aligned}}$$

* Multiplication is associative.

e.g.

$$(2 \times 3) \times 4 = 2 \times (3 \times 4)$$

$$\boxed{(ab)c = a(bc)}$$

* Commutative and associative properties together

When you multiply a lot of numbers, you can change the order of the numbers and form groups of numbers in parantheses as convenient as you like.

e.g. $25 \times 5 \times 125 \times 4 \times 2 \times 8$

$= (25 \times 4) \times (5 \times 2) \times (125 \times 8)$

$= 100 \times 10 \times 1000 = 1000000$

Note:

$$\boxed{\begin{array}{l} 2 \times 5 = 10 \\ 4 \times 25 = 100 \\ 8 \times 125 = 1000 \end{array}}$$

$(4 \times 25 = 2 \times 2 \times 5 \times 5)$

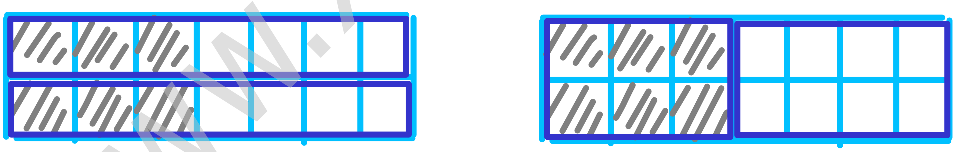
$(8 \times 125 = 2 \times 2 \times 2 \times 5 \times 5 \times 5)$

* Multiplying by 1 or 0.

$$\boxed{\begin{array}{l} 1 \cdot a = a \\ 0 \cdot a = 0 \end{array}}$$

* Multiplication distributes over addition.

e.g.



$2 \times (3+4) = 2 \times 3 + 2 \times 4$

$(3+4) \times 2 = 3 \times 2 + 4 \times 2$

distributing

$$\boxed{\begin{array}{l} a(b+c) = ab + ac \\ (b+c)a = ba + ca \end{array}}$$

factoring (out a common factor)

$$\boxed{\begin{array}{l} ab + ac = a(b+c) \\ ba + ca = (b+c)a \end{array}}$$

distributing $\longrightarrow$ right-hand side

$$\boxed{a(b+c+d+\cdots) = ab+ac+ad+\cdots}$$

left-hand side $\longleftarrow$ factoring

e.g. $17\cdot 13 + 51\cdot 13 + 32\cdot 13$

$= (17+51+32)\cdot 13$

$= (68+32)\cdot 13 = 100\cdot 13 = 1300$

* Practice

8. $25\times 125\times 9\times 4\times 8 =$ ______

9. $125\times 7\times 25\times 32 =$ ______

10. $51\times 9 + 51\times 31 =$ ______

11. $2024\times 2023\times 2022\times\cdots\times 3\times 2\times 1\times 0 =$ ______

12. The factorial of n is the product

$$n\cdot(n-1)\cdot(n-2)\cdots 3\cdot 2\cdot 1$$

e.g. $1! = 1$

$2! = 2\cdot 1 = 2$

$3! = 3\cdot 2\cdot 1 = 6$

(1) $6! =$ ______

(2) If $9! = 362880$, then $10! =$ ______

Negation

* negation

The negation of x (or called the opposite of x), written as $-x$, is the number that we add to x to get 0. That is

$$\boxed{-x + x = 0}$$

e.g.

$$\overbrace{-3}^{\text{opposite of } 3} + \underbrace{3}_{\text{opposite of } (-3)} = 0$$

The opposite of 3 is $-x = -3$.
The opposite of -3 is 3. That is

$$-(-3) = 3$$

* The opposite of the opposite of a number is itself.

Negation of negation

$$\boxed{-(-x) = x}$$

$$x \xrightarrow{\text{negation}} -x \xrightarrow{\text{negation}} x$$

$$\text{on} \xrightarrow[\text{Switch}]{\text{flip}} \text{off} \xrightarrow[\text{Switch}]{\text{flip}} \text{on}$$

* Multiplying by negation

e.g.

$$\begin{aligned}
4 \cdot 3 &= 12 \\
3 \cdot 3 &= 9 \\
2 \cdot 3 &= 6 \\
1 \cdot 3 &= 3 \\
0 \cdot 3 &= 0 \\
(-1) \cdot 3 &= \underline{-3}
\end{aligned}$$

+3
+3
+3
+3
+3

We define multiplication of x by (-1) as the negation of x.

$$\boxed{(-1) \cdot x = -x}$$

e.g. $(-2) \times 3 = (-1) \cdot 2 \cdot 3 = (-1) \cdot (2 \cdot 3) = -(2 \cdot 3)$

$2 \cdot (-3) = 2 \cdot (-1) \cdot 3 = (-1) \cdot (2 \cdot 3) = -(2 \cdot 3)$

$$\boxed{(-x) \cdot y = x \cdot (-y) = -(xy)}$$

e.g.

$$\begin{aligned}
(-2) \cdot (-3) &= (-1) \cdot 2 \cdot (-3) \\
&= -(2 \cdot (-3)) \\
&= -(-2 \cdot 3) \\
&= 2 \cdot 3
\end{aligned}$$

$$\boxed{(-x)(-y) = xy}$$

e.g. $(-1)(-2)(-3)(-4) = \big(\underbrace{(-1)\cdot(-2)}_{1\cdot 2}\big)\big(\underbrace{(-3)\cdot(-4)}_{3\cdot 4}\big) = 2\cdot 12 = 24$

$(-1)(-2)(-3) = \big(\underbrace{(-1)\cdot(-2)}_{1\cdot 2}\big)\cdot(-3) = 2\cdot(-3) = -6$

Product of even number of negative numbers is positive.
Product of odd number of negative numbers is negative.

e.g. $-(4+5) = (-1)\cdot(4+5)$

$= (-1)\cdot 4 + (-1)\cdot 5$

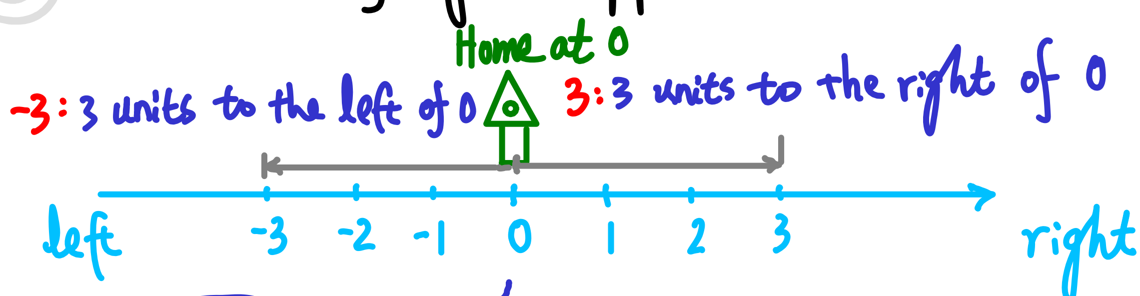
$= (-4) + (-5)$

$$\boxed{-(x+y) = (-x) + (-y)}$$

* negative numbers

e.g. You have an apple. The number of your apples is $\underline{1}$ or $\underline{+1}$.
You do not have any apple, and you owe someone an apple instead.
The number of your apples is -1.

e.g.



... negative numbers: left of 0 on the number line

* Practice

13. $-897 + (2024 + 897) =$ ________

14. $-387 + 388 =$ ________

15. $-388 + 387 =$ ________

16. $(-3) \times 25 =$ ________

17. $3 \times (-25) =$ ________

18. $(-3) \times (-25) =$ ________

19. $\underbrace{(-1)(-1)(-1)\cdots(-1)}_{2024\ (-1)\text{'s}} =$ ________

20. $\underbrace{(-1)(-1)(-1)\cdots(-1)}_{2025\ (-1)\text{'s}} =$ ________